Transverse Field Ising Spin Chain

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Just like the 2D classical Ising Model is the "Drosophila" of (thermal) phase transitions, we'll see that the transverse field Ising model is a prototypical example of a quantum phase transition. (a phase transition driven by quantum instead of thermal fluctuations)
↳ at temperature $T=0$

We'll show that in 1+1 d, this model is exactly soluble, as it can be mapped onto non-interacting ("free") Majorana fermions. In passing, we'll discuss a closely related model: the Kitaev chain, a model for a topological superconductor in one dimension.
In the next chapters, we will relate the quantum Ising chain to the classical Ising model: "Quantum-classical correspondence", and we will derive a field theory for the critical behavior near the quantum phase transition.

Ⓘ Quantum Ising Model : phases and excitations

ⓐ Ising Chain :

Degrees of freedom: spins $\frac{1}{2}$ (two level systems)

Hilbert space: $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$ ↑ # of spins

Hamiltonian:

$$\boxed{H = -J\sum_i \sigma_i^z\sigma_{i+1}^z + g\,\sigma_i^x}$$

Model of quantum magnetism

$\cdots\uparrow\cdots\downarrow\cdots\uparrow\cdots\uparrow\cdots$ 1d chain

- Here: $\sigma_i^\alpha = \overset{\begin{pmatrix}1&0\\0&1\end{pmatrix}}{\mathbb{1}} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes \sigma^\alpha \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1}$

$(\alpha = x, y, z)$

and $\sigma^z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\quad \sigma^x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad \sigma^y = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$

- This model is interesting because there's a competition between

- $\sigma_i^z \sigma_{i+1}^z$: favors aligned spins $\Rightarrow$ ferromagnetic interactions

- σ_i^x : favors spins pointing in x direction : $|\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$

eigenstate of σ^x

Ultimately, this competition arises because $[\sigma^x, \sigma^z] \neq 0$, but note that $H = -\sum_i g_x \sigma_i^x + g_z \sigma_i^z$ would instead be trivial

(Why? Think of $-\sum_i \vec{h} \cdot \vec{\sigma}_i$)

Ⓑ Symmetries : The transverse field Ising chain is invariant under a $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ symmetry generated by :

because of $h\sigma^x$

$$\boxed{\mathcal{C} = \prod_i \sigma_i^x}$$

$\mathcal{C}^2 = 1$:

$\mathcal{C}\,|\uparrow\uparrow\downarrow\ldots\rangle = |\downarrow\downarrow\uparrow\ldots\rangle$

flips all spins

We have $[H, \mathcal{C}] = 0$ using $\mathcal{C}\,\sigma_i^x\,\mathcal{C} = \sigma_i^x$

$\mathcal{C}\,\sigma_i^z\,\mathcal{C} = -\sigma_i^z$

$$\Rightarrow \quad \begin{aligned} H|\psi_n\rangle &= E_n|\psi_n\rangle \\ \mathcal{C}|\psi_n\rangle &= \pm|\psi_n\rangle \end{aligned}$$

(H, $\mathcal{C}$ can be diagonalized in the same basis)

ⓒ Phases and excitations

$g \gg 1$ Disordered Phase (Quantum paramagnet)

As $g \to \infty$, $H \simeq -Jg\sum_i \sigma_i^x$ which has a unique ground-state

$$|\psi_0\rangle = |\rightarrow\rightarrow\rightarrow\dots\rightarrow\rangle \quad \text{with} \quad |\rightarrow\rangle = \frac{|\uparrow\rangle + |\downarrow\rangle}{\sqrt{2}}$$

Clearly: $\mathcal{C}|\psi_0\rangle = |\psi_0\rangle$

In the z basis: $|\psi_0\rangle = \frac{1}{2^{N/2}}\sum_{\{\sigma\}} |\{\sigma\}\rangle$

superposition of all states

Excitations: $|i\rangle = |\rightarrow\rightarrow\dots\underset{i-1}{\rightarrow}\underset{i}{\leftarrow}\underset{i+1}{\rightarrow}\dots\rightarrow\rangle$

N such states, all degenerate if we neglect $-J\sum_i \sigma_i^z\sigma_{i+1}^z$.

- Energy gap $\Delta = 2Jg$ over $E_0 = -NJg$. (degenerate)
- Let's consider the effect of $\underbrace{-J\sum_i \sigma_i^z\sigma_{i+1}^z}_{V\,=\,\text{perturbation}}$ within perturbation theory.

$$\langle j|V|i\rangle = -J\left(\delta_{j,i-1} + \delta_{j,i+1}\right)$$

: V moves spin flips

$$H_{\text{eff}}|i\rangle = -J\left[|i-1\rangle + |i+1\rangle\right] + (E_0 + 2gJ)|i\rangle$$

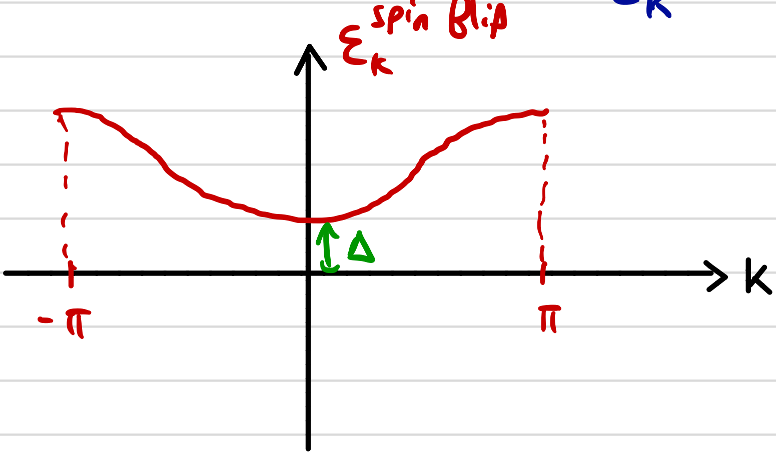
↑ work in that single spin flip basis (N×N matrix)

With periodic boundary conditions, H_{eff} can be diagonalized using Fourier transform: $|k\rangle = \frac{1}{\sqrt{N}}\sum_j e^{ikj}|j\rangle$, $|j\rangle = \frac{1}{\sqrt{N}}\sum_k e^{-ikj}|k\rangle$

$|0\rangle \equiv |N\rangle \Rightarrow kN = 2\pi n$ with $n = 0, 1, \ldots, N-1$

as $N \to \infty$, $k \in [-\pi, \pi]$ Brillouin zone

$$\Rightarrow (H - E_0)|k\rangle = \underbrace{(2Jg - 2J\cos k)}_{\varepsilon_k}|k\rangle$$



$\Delta = 2J(g-1)$ energy gap

$\varepsilon_k \simeq \Delta + Jk^2$ at small k

Quasiparticle excitations = spin flip

Extending (naively!!) this calculation valid for $g \gg 1$ to any g, we see that the gap for creating spin flips closes at $g_c = 1$.

$\boxed{g \ll 1:}$ Ordered phase (Ferromagnet)

Now in the opposite limit, $g \to 0$, $H \simeq -J\sum_i \sigma_i^z \sigma_{i+1}^z$

Two degenerate groundstates: $\left.\begin{aligned} |+\rangle &= |\uparrow\uparrow\cdots\uparrow\rangle \\ |-\rangle &= |\downarrow\downarrow\cdots\downarrow\rangle \end{aligned}\right\}$ Break the symmetry $\mathcal{E}|\pm\rangle = |\mp\rangle$

However, we need to be more carefull!

Ferromagnet: Spontaneous symmetry breaking

"Interlude": Spontaneous Symmetry Breaking (i)

We can form $|\psi_\pm\rangle = \frac{|+\rangle \pm |-\rangle}{\sqrt{2}}$

Aslo groundstates, but now preserve the symmetry.

Those are macroscopic cat states, but they respect the symmetry. The notion of GS preserving or not the symmetry seems arbitrary.

- To see what's going on, note that the degeneracy between $|\psi_\pm\rangle$ is lifted by the perturbation $V = -gJ\sum_i \sigma_i^x$. Note however that the degeneracy if lifted only at N^{th} order in perturbation theory!

$$\langle + | V^N | - \rangle = g^N \ll 1 \quad \text{but } \neq 0$$

$$H_{eff}^{\pm} = \begin{array}{cc} |+\rangle & |-\rangle \\ \end{array} \begin{pmatrix} E_0 & g^N \\ g^N & E_0 \end{pmatrix} \Rightarrow \text{eigenstates } |\psi_\pm\rangle = \frac{|+\rangle \pm |-\rangle}{\sqrt{2}}$$

with energy splitting $\delta = |E_+ - E_-|$
$= \mathcal{O}(g^N) \ll 1$

So for any finite size N, the true groundstate is $|\psi_+\rangle$, with an exponentially small splitting $\delta \sim e^{-N \log 1/g}$ with $|\psi_-\rangle$.

However: Suppose we prepare the system in the state $\underbrace{|\uparrow\uparrow\dots\uparrow\rangle = |+\rangle}_{\text{not a true eigenstate}}$.

$$|\psi(t)\rangle = \frac{e^{-iE_+t}|\psi_+\rangle + e^{-iE_-t}|\psi_-\rangle}{\sqrt{2}}$$

Proba to find the system in $|+\rangle$ state at time t:

$$P(t) = |\langle + | \psi(t)\rangle|^2 = \cos^2\left[\frac{\delta t}{2}\right] \boxed{\simeq 1} \text{ untill } t \ll 1/\delta \sim e^{N \log 1/g}$$

"Interlude": Spontaneous Symmetry Breaking (ii)

So for $N \sim 10^{23}$ spins, $|+\rangle$ is essentially a "true" GS for all purposes, unless one is willing to wait $t \sim e^{10^{23}}$ to see a tunelling process to $|-\rangle$. "Spontaneous symmetry breaking"

More formally: let's add a small longitudinal field $-h_z \sum_i \sigma_i^z$ with h_z small, but larger than δ (automatically satisfied for $N \gg 1$)

$\Rightarrow$ this will "select" $|+\rangle$ as the true groundstate (Note that h_z breaks $\mathcal{E}$!)

$$\boxed{\lim_{h_z \to 0} \lim_{N \to \infty} \langle \sigma_i^z \rangle \neq 0}$$

in ferromagnetic phase

BROKEN SYMMETRY

order parameter $= \langle GS | \sigma_i^z | GS \rangle$

$\mathcal{E} \sigma_i^z \mathcal{E} = - \sigma_i^z$ "odd"

⚠ Order of limit: $\lim_{N \to \infty} \lim_{h_z \to 0} \langle \sigma_i^z \rangle = 0$ everywhere by symmetry

Another way to diagnose spontaneous symmetry breaking without introducing a small (symmetry breaking) field is to look for long range order in correlation functions: (now $h_z = 0$, $|GS\rangle = |\psi_+\rangle$)

$$\langle GS | \sigma_i^z \sigma_j^z | GS \rangle \neq 0 \quad \text{even as } |i-j| \to \infty$$

$\left(= 1 \text{ if } g = 0 \right)$

in particular, note that $\langle \sigma_i^z \sigma_j^z \rangle \neq \langle \sigma_i^z \rangle \langle \sigma_j^z \rangle = 0$ even for very distant spins i, j!

$\Rightarrow$ ordered phase. Note: this second definition is OK even for the true (cat state, symmetry preserving) groundstate

$$\langle \sigma_i^z \sigma_j^z \rangle = \frac{1}{2} \left(\langle + | \sigma_i^z \sigma_j^z | + \rangle + \langle - | \sigma_i^z \sigma_j^z | - \rangle \right) = 1$$

Excitations: spin flip: No! $\uparrow\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow$ Energy cost $\Delta E = 4J$

Domain wall ("kink"): $|\bar{i}\rangle = \uparrow\uparrow\uparrow\underset{i}{\uparrow}\,\underset{i+1}{\downarrow}\downarrow\downarrow\downarrow\downarrow$ $\Delta E = 2J$

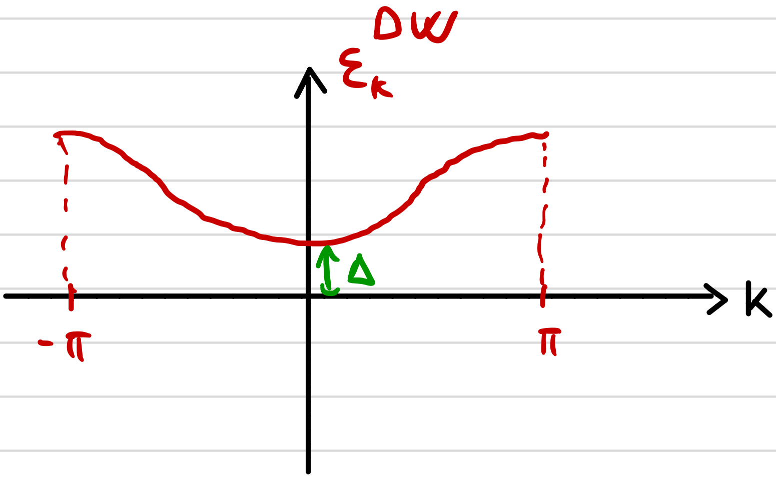
Note: with PBC, DW's come in pairs, but they're really independent excitations.

Perturbation theory for $g \ll 1$: $-Jg\sum_i \sigma_i^x$ moves DW

$|\bar{i}\rangle$: DW between i and $i+1$ (position: $\bar{i} = i + 1/2$)

$$(H - E_0)|\bar{i}\rangle = -gJ\left(|\bar{i}+1\rangle + |\bar{i}-1\rangle\right) + 2J|\bar{i}\rangle$$

$$\Rightarrow \varepsilon_k = 2J(1 - g\cos k)$$



$\Delta = 2J(1-g)$ energy gap

$\varepsilon_k \simeq \Delta + Jk^2$ at small k

Quasiparticle excitations = DW

- Again, if we close our eyes and extend this result away from $g \ll 1$ we see that DW excitations become gapless at $g = g_c = 1$.
- Note that these results suggest a duality between the two phases:

$$\boxed{g \longleftrightarrow J_g}$$

We're going to make this more precise in the next section.

⓶ Kramers - Wannier Duality:

Let's make this duality between the two phases more precise:

- let's start from the FM phase. Excitations = DW :

- Map: spin configuration $\Longleftrightarrow$ DW configuration (2 to 1!)

Define: DW variable $\tau_{\bar{i}}^{x} = \sigma_i^z \sigma_{i+1}^z = \begin{cases} +1 & \text{if no DW} \\ -1 & \text{if DW} \end{cases}$ Between i and i+1

$i = \bar{i} - 1/2$

Note: $(\tau_{\bar{i}}^{x})^2 = 1$ and $\tau^{x\dagger} = \tau^x$

Create a DW: $\tau_{\bar{i}}^{z} = \prod_{i > \bar{i}} \sigma_i^x$: flip all spins to the right to $\bar{i}$

Now: $\tau_{\bar{i}}^{z} \tau_{\bar{j}}^{x} = \tau_{\bar{j}}^{x} \tau_{\bar{i}}^{z}$ if $\bar{i} \neq \bar{j}$ (if $\bar{i} > \bar{j}$: don't share any site
if $\bar{i} < \bar{j}$: share two sites)

and $\tau_{\bar{i}}^{z} \tau_{\bar{i}}^{x} = - \tau_{\bar{i}}^{x} \tau_{\bar{i}}^{z}$ (share one site: $\sigma_{i+1}^{x} \sigma_{i+1}^{z} = - \sigma_{i+1}^{z} \sigma_{i+1}^{x}$)

$\Rightarrow$ Pauli matrices algebra and $(\tau^{x,z})^2 = 1$

Note that $\tau_{\bar{i}-1}^{z} \tau_{\bar{i}}^{z} = \sigma_i^x$ local

$$\Rightarrow \boxed{\begin{aligned} H &= -J \sum_i \sigma_i^z \sigma_{i+1}^z + g\, \sigma_i^x \\ &= -J \sum_{\bar{i}} \tau_{\bar{i}}^{x} + g\, \tau_{\bar{i}}^{z} \tau_{\bar{i}+1}^{z} \end{aligned}}$$

Same Hamiltonian using these "dual" DW variables, but different couplings: $g \to 1/g$ $J \to Jg$

Remarks:

- The FM phase of the τ spins corresponds to the PM phase of the σ spins, and vice versa.

- The two phases are different: PM of τ's has unique GS, which corresponds to the FM phase of the original σ spins, which we know has a doubly degenerate GS (for $N \to \infty$). This is because this DW description is a 2-to-1 mapping!

  Similarly, the FM phase of the τ spins naively has a doubly degenerate GS, which we know is incorrect since it corresponds to the PM phase of the σ spins, which has a unique GS. (This can be fixed by treating the boundary conditions more carefully, or by fixing $\lim_{i \to +\infty} \tau_i^z = 1$ $= +1$)

- Fixes critical coupling to: $\boxed{g_c = 1}$ (self dual point)

Physical picture:

PM phase τ^z has an expectation value $\langle GS | \tau_i^z | GS \rangle \underset{g\to\infty}{=} 1$

$\Rightarrow$ DW "condensate"
$$|GS\rangle_{g=\infty} = \frac{1}{2^{N/2}} \sum_{\text{all states}} |\{\sigma^z\}\rangle$$

$\langle \sigma_i^z \sigma_j^z \rangle$ decays quickly (exponentially) since this correlator flips sign each time a DW intervenes between i and j

FM phase $\langle \sigma_i^z \rangle \neq 0$ ordered phase

$\uparrow$ with caveats mentioned above!

DW: $\langle \tau_i^z \rangle = 0$ DW are massive excitations

$\boxed{g = g_c}$ Quantum phase transition. Natural variable = Majorana Fermions!

Ⅲ Majorana Fermions and exact solution

We've seen that the two phases are described by different variables:

- PM $(g \gg 1)$: GS = DW condensate $(\langle \tau_{\bar{i}}^z \rangle \neq 0)$, excitations = spin flips σ_i^z
- FM $(g \ll 1)$: GS = spin condensate $(\langle \sigma_i^z \rangle \neq 0)$, excitations = DW $\tau_{\bar{i}}^z$
  (a.k.a. Ferromagnet!)

"Correct" variables right at the transition?

$\Rightarrow$ "Attach spin to a domain wall" Correct variables $\forall g$

Let:

$$\boxed{\begin{aligned} a_i &= \sigma_i^z \tau_{\bar{i}}^z = \sigma_i^z \prod_{j > \bar{i}} \sigma_j^x \\ b_i &= \sigma_i^y \tau_{\bar{i}}^z \end{aligned}}$$

$(\bar{i} = i + 1/2)$

Jordan Wigner transformation (more on this later)

- If $g \gg 1$: $\tau_{\bar{i}}^z \simeq \langle \tau_{\bar{i}}^z \rangle \simeq 1 \Rightarrow a_i \simeq \sigma_i^z$
  $g \ll 1$: $\sigma_i^z \simeq \langle \sigma_i^z \rangle \simeq 1 \Rightarrow a_i \simeq \tau_{\bar{i}}^z$

  excitation in both phases!

- Algebraic properties:
  - $a_i^\dagger = a_i$, $b_i^\dagger = b_i$ (real) and $\{a_i, b_i\} = 0$, $a_i^2 = b_i^2 = 1$
  - a_i and b_i are fermions: $\{a_i, a_j\} = 0$ $(i \neq j)$
    (non-locality comes from "string" $\prod \sigma^x$) $\{b_i, b_j\} = 0$, $\{a_i, b_j\} = 0$

This follows since the spin flip σ_i^z (or σ_i^y) changes sign as it moves through the DW created by $\tau_{\bar{j}}^z$ (with $j < i$)

Note: | DW + charge = fermion in 1d
| in higher d: attach charge to vortex ⇒ anyons

$$\boxed{\begin{array}{l}\{a_i, b_j\} = 0\\ \{a_i, a_j\} = 2\delta_{ij}\\ \{b_i, b_j\} = 2\delta_{ij}\end{array}}$$

⇒ Majorana (real) fermions!

$$\left.\begin{array}{l} c_j^\dagger = \frac{1}{2}(a_j + i b_j)\\ c_j = \frac{1}{2}(a_j - i b_j)\end{array}\right\} \text{Usual (Dirac) fermions}$$

$$\{c_i^\dagger, c_j\} = \delta_{ij} \qquad \begin{array}{l}\{c_i, c_j\} = 0\\ \{c_i^\dagger, c_j^\dagger\} = 0\end{array}$$

Remarkably, these fermionic variables will allow us to naturally describe the quantum critical point ⇒ "Emerging Majorana Fermions"
In fact, this description will lead to an Exact Solution!

- Express spins in terms of fermions:

$$a_i b_i = \sigma_i^z \tau_i^z \sigma_i^y \tau_i^z = \underbrace{\sigma_i^z \sigma_i^y}_{-i\sigma_i^x} \underbrace{(\tau_i^z)^2}_{1} = -i\sigma_i^x$$

and
$$a_i b_i = (c_i^\dagger + c_i)\frac{(c_i^\dagger - c_i)}{i} = +i(c_i^\dagger c_i - c_i c_i^\dagger)$$
$$= i(2n_i - 1) \quad \text{with } n_i = c_i^\dagger c_i$$
$$= -i(-1)^{n_i}$$

$$\Rightarrow \sigma_j^x = -i a_j b_j = (-1)^{n_j}$$
$$= 1 - 2n_j$$

$|\rightarrow\rangle \Leftrightarrow |0\rangle$
$|\leftarrow\rangle \Leftrightarrow |1\rangle$
Spin (x-basis) Fermions

$$a_i b_{i+1} = \sigma_i^z \tau_i^z \sigma_{i+1}^y \tau_{i+1}^z = \sigma_i^z \underbrace{\sigma_{i+1}^x \sigma_{i+1}^y}_{i\sigma_{i+1}^z} = i\sigma_i^z \sigma_{i+1}^z$$

and
$$a_j b_{j+1} = (c_j^\dagger + c_j)\frac{(c_{j+1}^\dagger - c_{j+1})}{i} = -i(c_j^\dagger c_{j+1}^\dagger - c_j^\dagger c_{j+1}$$
$$+ c_j c_{j+1}^\dagger - c_j c_{j+1})$$

$$\Rightarrow \quad \sigma_j^z \sigma_{j+1}^z = -i\, a_j b_{j+1} = c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1}$$

We can therefore rewrite the Hamiltonian as:

$$\boxed{\begin{aligned} H &= -J \sum_j \sigma_j^z \sigma_{j+1}^z + g\, \sigma_j^x \\ &= +Ji \sum_j \left(a_j b_{j+1} + g\, a_j b_j \right) \\ &= -J \sum_j \Big(c_{j+1}^\dagger c_j + c_{j+1}^\dagger c_j^\dagger + h.c. \\ &\qquad\qquad - 2g\, n_i \Big) + cst \end{aligned}}$$

spins

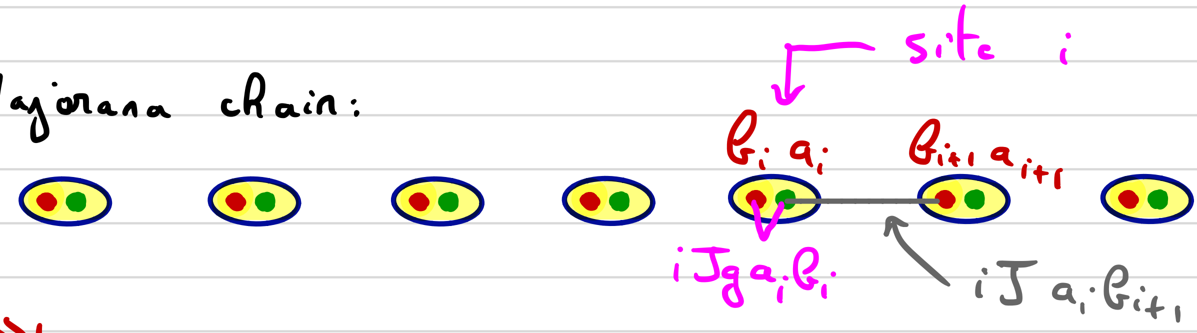
Majorana fermions

Dirac fermions

- Crucially, the Ising model is quadratic (non-interacting) in terms of fermions $\Rightarrow$ Exact solution!
- In the fermionic language: $\sigma^x \Leftrightarrow$ chemical potential

  $\sigma^z \sigma^z$: hopping + mean-field-like superconducting term $c_{i+1}^\dagger c_i^\dagger$

Majorana chain:



$g \gg 1$:

Fermion $c_i^\dagger = \frac{a_i + i b_i}{2}$ in vacuum state $\quad c_i |0\rangle = 0$

$n_i = 0 \quad \forall i$

$g = 0$:

GS = vacuum of these new fermions $\quad \tilde{c}_j^{+} = \frac{a_j - i b_{j+1}}{2}$

BUT: with free boundary conditions, there are two free Majorana edge modes:

$$[H, b_0] = [H, a_N] = 0 \quad (N \text{ sites})$$

(zero modes: don't change the energy): we can form a Dirac fermion:

$$d^{+} = \frac{b_0 + i a_N}{2}$$

this fermion state can be occupied or not in the GS ⇒ 2 degenerate groundstates

In the bulk, the two phases look identical, but the FM (symmetry-broken) phase for the spins maps onto a topological phase of the fermions with no order parameter, and 2 Majorana edge modes.

Note: • the fermions in the Ising chain are purely formal, but some 1d superconducting wires can realize a similar topological fermionic phase, where the fermions are now physical (see Kitaev chain below)

• The edge modes survive for $g < g_c = 1$ (gapped phase)

Exact Solution : Let's work with the Dirac fermions c_i^{+} since they're a bit more familiar. Use Fourier transform:

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{-ikj} \qquad \left(c_j = \frac{1}{\sqrt{N}} \sum_k c_k e^{ikj} \right)$$

$$c_N = c_0 \qquad k = \frac{2\pi}{N} n$$

$$\sum_j c_{j+1}^\dagger c_j + h.c. = \overbrace{\frac{1}{N}\sum_j \sum_{k,k'} e^{ij(k-k')}}^{=\delta_{k,k'}} c_k^\dagger c_{k'} \underbrace{\left(e^{-ik} + h.c.\right)}_{2\cos k}$$

$$= \sum_k 2\cos k \; n_k$$

$$\sum_j c_{j+1}^\dagger c_j^\dagger + h.c. = \overbrace{\frac{1}{N}\sum_j \sum_{k,k'} e^{ij(k+k')}}^{\delta_{k,-k'}} c_k^\dagger c_{k'}^\dagger e^{-ik} + h.c.$$

$$c_k^\dagger c_{-k}^\dagger = -c_{-k}^\dagger c_k^\dagger \Rightarrow e^{-ik} \to \frac{e^{-ik} - e^{ik}}{2}$$

$$= \sum_k +i\sin k \left(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k\right)$$

$$\Rightarrow H = J\sum_k 2(g-\cos k)\, n_k - i\sin k \underbrace{\left(c_{-k}^\dagger c_k^\dagger + c_{-k} c_k\right)}_{\text{breaks fermion number } \left(\text{mod } 2\right)} + cst$$

(not quite diagonal: mixes k and -k modes)

Let $\Psi_k = \begin{pmatrix} c_{-k} \\ c_k^\dagger \end{pmatrix}$ $\quad \Psi_k^\dagger = \begin{pmatrix} c_{-k}^\dagger \\ c_k \end{pmatrix}^T = \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_{-k}\right]^T$

not independent

$$\Rightarrow H = J\sum_k \Psi_k^\dagger \begin{pmatrix} g-\cos k & -i\sin k \\ i\sin k & -g+\cos k \end{pmatrix} \Psi_k$$

Eigenmodes: $\Psi_k = U\Omega_k$ s.t. $U^\dagger J\begin{pmatrix} g-\cos k & -i\sin k \\ i\sin k & -g+\cos k \end{pmatrix} U = \begin{pmatrix} \frac{\varepsilon_k}{2} & 0 \\ 0 & -\frac{\varepsilon_k}{2} \end{pmatrix}$

with $\boxed{\varepsilon_k = 2J\sqrt{(\cos k - g)^2 + \sin^2 k}}$ "Bogoliubov transformation"

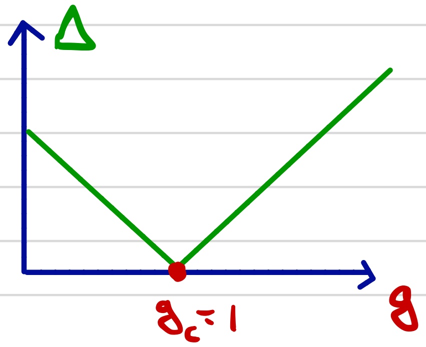
$\Omega_k^\dagger = \begin{pmatrix} d_{-k}^\dagger \\ d_k \end{pmatrix}^T \Rightarrow \boxed{H = \sum_k \varepsilon_k \left(d_k^\dagger d_k - \frac{1}{2}\right) + cst}$ Normal modes

excitations over GS. $\quad d_k|GS\rangle = 0 \;\; \forall k$

U = unitary transformation, preserves (anti)-commutation relations. d_k: fermion

First excited state: $d_0^\dagger|GS\rangle$, with energy gap $\boxed{\Delta = 2J|g-1|}$

Miraculously, this result valid for any g coincides with our estimates from first order perturbation theory (!!!).



At $g = g_c = 1$, the spectrum becomes gapless

Low energy spectrum: (small k, $|g-1| \ll 1$)

$$\varepsilon_k = \underbrace{2J}_{c\,=\,\text{"speed of light"}} \sqrt{\underbrace{(g-1)^2}_{\sim\, m^2 = \xi^{-2}} + k^2}$$

ξ = correlation length

Emergent Lorentz symmetry in the spectrum!

At the critical point: $\boxed{\varepsilon_k = c|k|}$ gapless spectrum ($\varepsilon_k \to 0$ as $k \to 0$)

$$v_k = \partial_k \varepsilon_k = \pm c$$

Critical Exponents:

- $\xi \sim |g-g_c|^{-\nu}$ with $\nu = 1$: Diverging correlation length
- $\varepsilon_k \sim k^z$, $z = 1$ dynamical critical exponent

Symmetry: $t \to \lambda^z t$, $x \to \lambda x$ $z=1$ for relativistic theory (time and space equivalent)

We'll make this symmetry more explicit by studying the continuum limit of this model in another chapter. This will be our first example of how a Quantum Field Theory emerges near a quantum phase transition.

IV Kitaev Chain and Majorana edge modes

In the Ising chain, we've shown that the FM (symmetry-broken) phase was equivalent, up to a Jordan Wigner transformation, to a fermionic phase with Majorana edge modes. Can we realize this phase using physical electrons?

Yes: Kitaev chain (1d superconducting wire, spinless electrons)

$$\boxed{H = \underbrace{-t\sum_i \left(c_{i+1}^\dagger c_i + h.c.\right)}_{\text{hopping}} \underbrace{-\mu \sum_i c_i^\dagger c_i}_{\text{chemical potential}} + \underbrace{\Delta \sum_i \left(c_{i+1}^\dagger c_i^\dagger + h.c.\right)}_{\substack{\text{mean-field superconducting term} \\ (\text{"}p\text{-wave": spinless } e^-)}}}$$

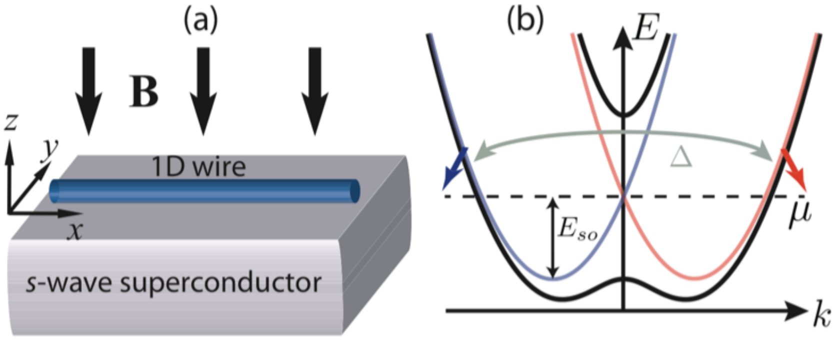
This Hamiltonian can be diagonalized (HW #1), and has a phase transition for $\mu = 2J$.

- For $\mu < 2J$: Topological Superconductor phase with Majorana edge modes.
- 

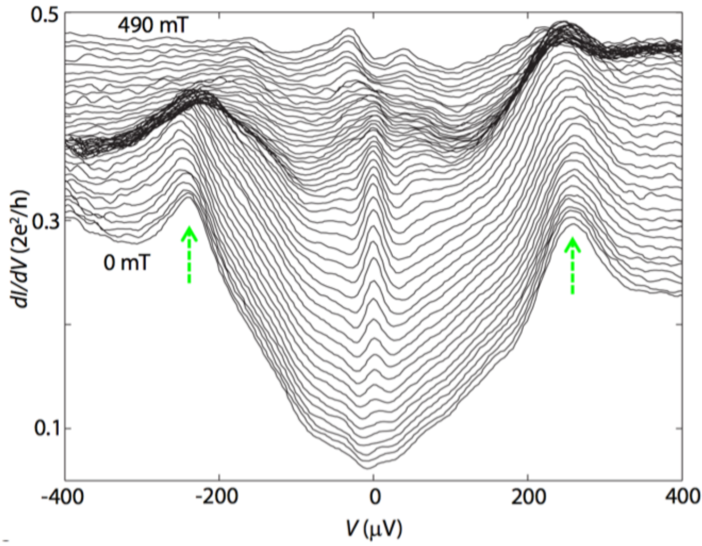
  ↳ zero energy
  Majorana zero modes localized near the end of the wire.
- Majorana = 1/2 of an electron (fractionalization)
- Example of topological phase (no order parameter, edge states)
- Form a qubit: $|\uparrow\rangle = d^\dagger |0\rangle$ $\quad d^\dagger = \dfrac{a+ib}{2}$
  $|\downarrow\rangle = |0\rangle$

In principle, protected from decoherence (non local object, protected by topology). In practice: • finite wire
• $T > 0$: excitations in the bulk

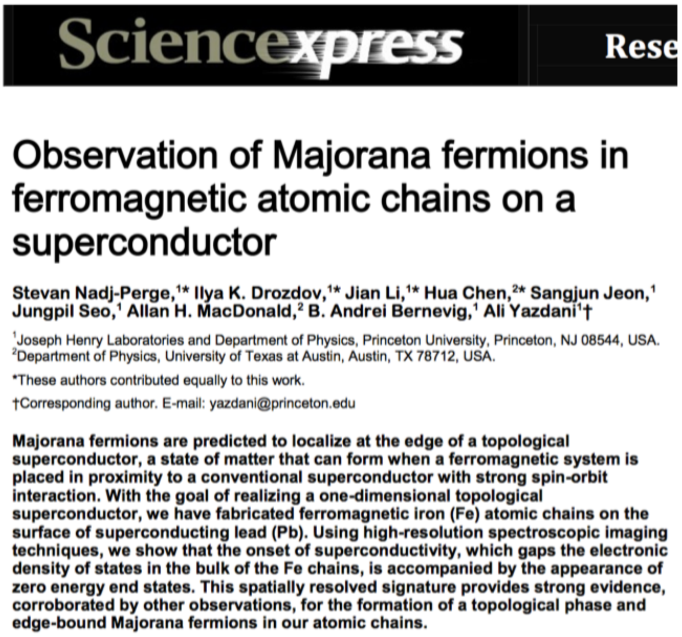
Can be realized using a spin-orbit coupled wire + proximity-induced superconductivity + external magnetic field.



(Figures from Alicea '12)



Mourik et al, Science '12 (Delft group)


Sciencexpress Rese

Observation of Majorana fermions in ferromagnetic atomic chains on a superconductor

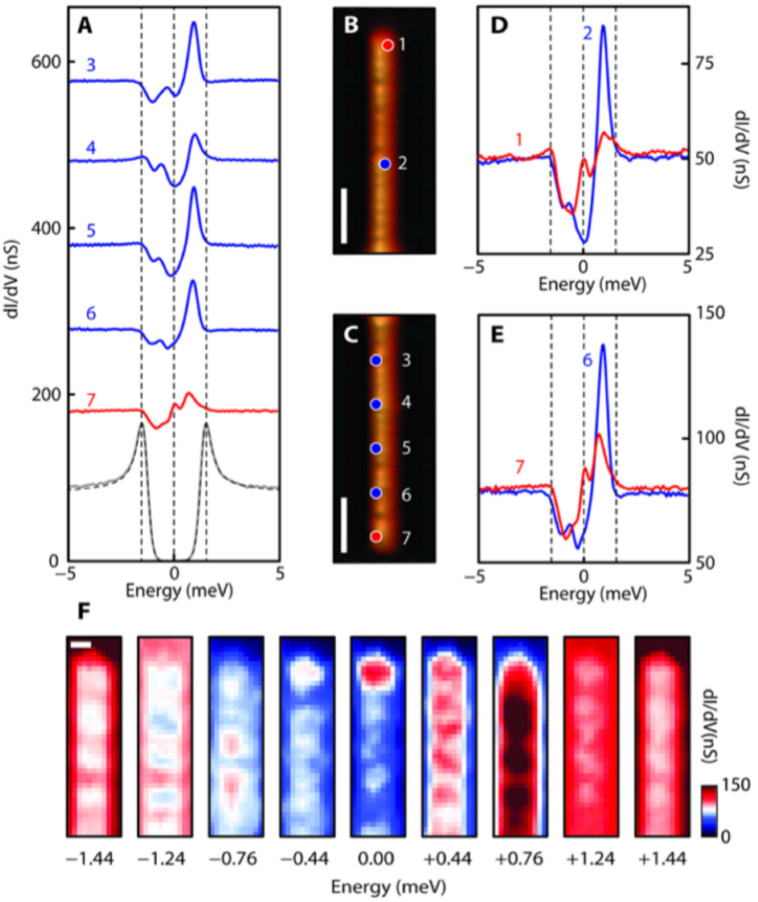
Stevan Nadj-Perge,[1]* Ilya K. Drozdov,[1]* Jian Li,[1]* Hua Chen,[2]* Sangjun Jeon,[1] Jungpil Seo,[1] Allan H. MacDonald,[2] B. Andrei Bernevig,[1] Ali Yazdani[1]†

[1]Joseph Henry Laboratories and Department of Physics, Princeton University, Princeton, NJ 08544, USA. [2]Department of Physics, University of Texas at Austin, Austin, TX 78712, USA.

*These authors contributed equally to this work.

†Corresponding author. E-mail: yazdani@princeton.edu

Majorana fermions are predicted to localize at the edge of a topological superconductor, a state of matter that can form when a ferromagnetic system is placed in proximity to a conventional superconductor with strong spin-orbit interaction. With the goal of realizing a one-dimensional topological superconductor, we have fabricated ferromagnetic iron (Fe) atomic chains on the surface of superconducting lead (Pb). Using high-resolution spectroscopic imaging techniques, we show that the onset of superconductivity, which gaps the electronic density of states in the bulk of the Fe chains, is accompanied by the appearance of zero energy end states. This spatially resolved signature provides strong evidence, corroborated by other observations, for the formation of a topological phase and edge-bound Majorana fermions in our atomic chains.



Nadj-Perge et al, Science '14 (Princeton group)

Dirac Hamiltonian and Majorana zero modes (HW)

Near the transition, the Kitaev chain is described by a Dirac Hamiltonian:

$$H = \frac{1}{2}\left(c_1^\dagger \dots c_N^\dagger\ c_1 \dots c_N\right) H_{BdG} \begin{pmatrix} c_1 \\ \vdots \\ c_N \\ c_1^\dagger \\ \vdots \\ c_N^\dagger \end{pmatrix}$$

$2N\times 2N$ matrix = Bogoliubov De Gennes Hamiltonian

in k space : $H = \sum_k \left(-\mu - 2t\cos k\right) c_k^\dagger c_k + \Delta \sum_k 2i\sin k \left(c_k^\dagger c_{-k}^\dagger + h.c.\right)$

$$= \frac{1}{2}\sum_k \left(c_k^\dagger\ c_{-k}\right) H_{BdG} \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix}$$

$$\Rightarrow \boxed{H_{BdG}(k) = \left(-2t\cos k - \mu\right)\tau_z + 2\Delta \sin k\, \tau_y}$$

(2×2 matrix)

Pauli matrices in BdG space

Spectrum: $E = \pm\sqrt{(2t\cos k + \mu)^2 + 4\Delta^2 \sin^2 k}$

linearize at small k for gap closing near $\mu = -2t$:

$$E = \pm\sqrt{m^2 + 4\Delta^2 k^2}$$

Dirac spectrum $\quad m = -\mu - 2t$

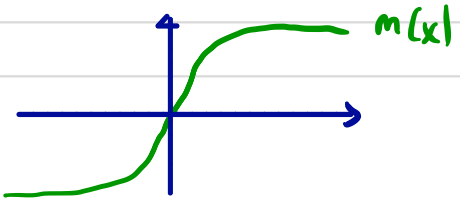
$m < 0$ topological phase

$m >$ trivial phase

$$H_{BdG}(k) \simeq m\,\tau_z + 2\Delta k\,\tau_y$$

$$\Rightarrow \quad H_{BdG} \simeq m\,\tau_z - 2\Delta i\,\partial_x \tau_y$$

Majorana zero mode : Imagine a Domain Wall in space



Let's look for a zero-mode: $H\Psi = 0$ $\qquad \Psi = \begin{pmatrix}\psi \\ \psi^\dagger\end{pmatrix}$

$$\Rightarrow \quad i\tau_y \, 2\Delta \, \partial_x \Psi(x) = m(x)\,\tau_z \Psi(x)$$

$$\Rightarrow \quad \partial_x \Psi(x) = \frac{m(x)}{2\Delta}\,\tau_x \Psi(x)$$

Solutions: $\Psi(x) = C\exp\left(\tau_x \int_0^x dx\, \frac{m(x)}{2\Delta}\right)\Psi(0)$

linearly independent solutions: $\Psi(x) = C\exp\left(\underbrace{\pm}_{\text{eigenvalues of } \tau_x} \int_0^x \frac{m(x)}{2\Delta}\right)\begin{pmatrix}1 \\ \pm 1\end{pmatrix}$

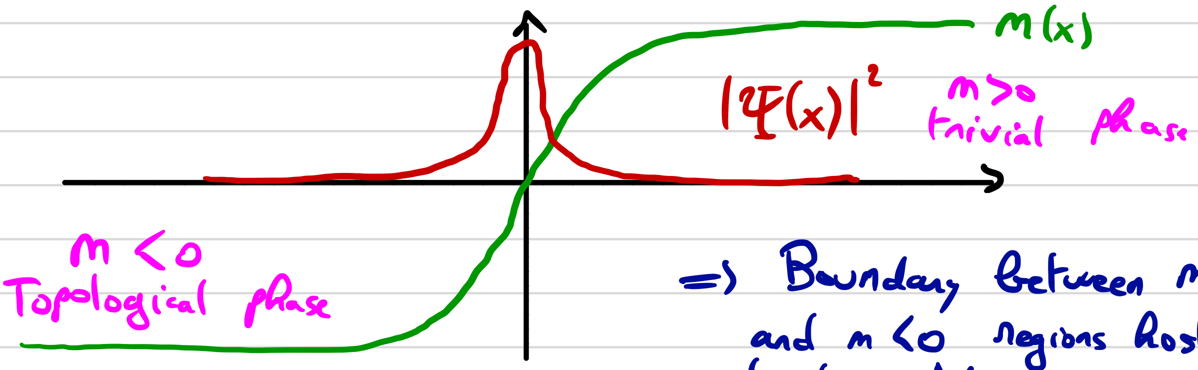
(diagonalize τ_x)

normalizable only if m changes sign!

$\Rightarrow$ zero mode $\boxed{\Psi(x) = C\exp\left(-\int_0^x dx\, \frac{m(x)}{2\Delta}\right)\begin{pmatrix}1 \\ -1\end{pmatrix}}$

Majorana zero mode (real solution)

exponentially localized near Domain Wall



$\Rightarrow$ Boundary between $m>0$ and $m<0$ regions hosts a localized Majorana zero mode!

Microscopically: