Scale Invariance and the RG

Scale invariance and a brief Introduction to the Renormalization Group

In this chapter, we'll study some of the scaling (critical) properties of the Ising model using the field theory description:

$$\boxed{S = \frac{1}{2}\int dz\, d\bar{z}\left[\chi_+ \partial \chi_+ - \chi_- \bar{\partial}\chi_- + i\Delta \chi_+ \chi_-\right]}$$

that we derived in the previous chapter. In the process, we'll discuss the symmetries of the action at criticality ($\Delta = 0$): scale and conformal invariance. We'll also introduce some basics of the Renormalization group to introduce scaling dimensions and critical exponents. This analysis will also justify the gradient expansion of the previous chapter, and explain why the critical properties of the Majorana Theory apply to all spin systems with a $\mathbb{Z}_2$ symmetry (in 1+1d $\Leftrightarrow$ classical 2d) going beyond the exactly solvable case of the Ising model.

$\Longrightarrow$ Universality

⚠ This chapter is <u>NOT</u> meant to be a thorough intro-duction to the Renormalization Group (RG). If you're not familiar with the idea of RG in stat. mech. and condensed mat-ter, this chapter will give you at best a vague introduction, and I recommend that you take P817 next year (which is basi-cally entirely dedicated to the RG: ϕ^4 theory, ε-expansion etc.). If you took that course (or equivalent), this chapter should be a good reminder.

(I) Scale invariance, Conformal invariance and correlation functions

- **Scale invariance**: Consider the $T=0$, critical action $\Delta = 0$

$$\mathcal{L} = \frac{1}{2}\left(-X_-\bar{\partial}X_- + X_+\partial X_+\right) \quad \text{and} \quad S = \int dz\, d\bar{z}\, \mathcal{L}$$

recall that $X \sim 1/\sqrt{a}$

It's invariant under rescaling: $x \to \lambda x$, $\tau \to \lambda \tau$ $\quad X_\pm \to \lambda^{-1/2} X_\pm$

$z = \lambda w$
$\partial_z = \lambda^{-1}\partial_w$
$d_z = \lambda\, dw$

$$\Rightarrow \quad S = \frac{1}{2}\int dw\, d\bar{w}\left(-\tilde{X}_-\bar{\partial}_w\tilde{X}_- + \tilde{X}_+\partial_w\tilde{X}_+\right)$$

with $\tilde{X}_\pm = \sqrt{\lambda}\, X_\pm$

Focus on X_-: $\langle X_-(z_1)\, X_-(z_2)\rangle = \langle X(z_1 - z_2)\, X(0)\rangle$ ← translation

$$= \lambda^{-1}\left\langle \tilde{X}_-\left(\frac{z_1 - z_2}{\lambda}\right)\tilde{X}_-(0)\right\rangle$$

$$\Rightarrow \quad \boxed{\langle X_-(z)\, X_-(0)\rangle = \frac{C}{z}}$$

← unimportant constant.

$z = x + i\tau$
$\;\; = x - t$

Note: Of course, we could've computed this correlation function directly in the path integral formalism as a Gaussian integral:
↑ Grassmann

$$\langle X_-(z)\, X_-(0)\rangle \sim \left[\bar{\partial}\right]^{-1} \sim \frac{1}{z} \qquad \text{since} \quad \bar{\partial}\left(\frac{1}{z}\right) = \pi\,\delta(z,\bar{z})$$

- **Conformal invariance**: In fact, the action is invariant under an infinitely large group of symmetries called conformal group (preserve angles in 2d)

$w = f(z)$
$\bar{w} = g(\bar{z})$

$$\partial_w = \frac{\partial z}{\partial w}\partial_z \qquad dw\, d\bar{w} = \frac{\partial w}{\partial z}\frac{\partial \bar{w}}{\partial \bar{z}}\, dz\, d\bar{z}$$

dropped $\tilde{\chi}$

$$\chi_-(w) = \left(\frac{\partial z}{\partial w}\right)^{1/2} \chi_-(z) \qquad \chi_+(\bar{w}) = \left(\frac{\partial \bar{z}}{\partial \bar{w}}\right)^{1/2} \chi_+(\bar{z})$$

In general: Primary fields : $\boxed{\phi(w,\bar{w}) = \left(\frac{\partial w}{\partial z}\right)^{-h} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z,\bar{z})}$

$(h,\bar{h})$ = conformal weights

ex: $\chi_- : (h,\bar{h}) = (1/2, 0)$

$\chi_+ : (h,\bar{h}) = (0, 1/2)$

$\underbrace{\chi_+ \chi_-}$: $\left(\frac{1}{2}, \frac{1}{2}\right)$

called energy/thermal operator. Drives the transition.

Under a scale transformation: $w = z/\lambda$

$$\Rightarrow \phi(w,\bar{w}) = \lambda^{\Delta} \phi(z,\bar{z}) \quad \text{with} \quad \boxed{\Delta = h + \bar{h}}$$

Scaling Dimension

Rotation: $w = e^{i\phi} z \Rightarrow \phi(w,\bar{w}) = e^{i\phi(\bar{h}-h)} \phi(z,\bar{z})$

$s = h - \bar{h}$: "conformal spin." (Not crucial for us)

Two point function: $\langle \phi(z,\bar{z}) \phi(0,0) \rangle = \dfrac{C}{z^{2h} \bar{z}^{2\bar{h}}}$ Fixed by conformal invariance

For fields with $s = 0$ $(h = \bar{h})$ called scalar operator:

$$\boxed{\langle \phi(r) \phi(0) \rangle = \frac{C}{r^{2\Delta}}} \qquad \text{with } z\bar{z} = |z|^2 = r^2$$

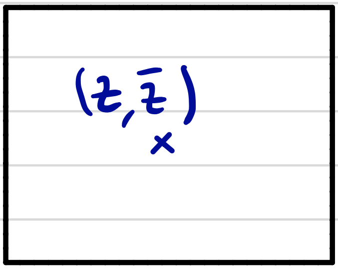
$\Delta = 2h$

- Convince yourself that if ϕ is a primary field, derivatives like $\partial^n \phi$ are not primary fields. Those are called "descendants" and they still transform as $\tilde{\phi} = \lambda^{\Delta_n} \phi$ under scale transfo. with $\Delta_n = \Delta + n$

What is this formalism good for?

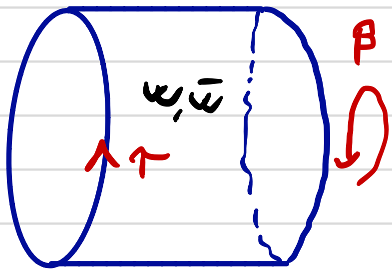
Let's say we now want to compute a correlation function at finite temperature T. This means we should consider the theory on a cylinder with (anti) periodic boundary conditions in imaginary time. In principle, we could do this calculation using discrete sums over Matsubara frequencies, but there's a much more elegant way to proceed:

Conformal mapping:



$$w = \frac{\beta}{2\pi} \log z \longrightarrow$$

$$z = e^{\frac{2\pi}{\beta} w}$$



Complex plane $\mathbb{C} \setminus \{0\}$ "$T = 0$" Cylinder "Finite T"

$w = x + i\tau$

$\tau \in [0, \beta]$

$$z = |z| e^{i\theta} \Rightarrow |z| = e^{2\pi/\beta\, x} \qquad \theta = \frac{2\pi}{\beta} \tau$$

on cylinder T=0 result

$$\langle X_-(w_1) X_-(w_2) \rangle_\beta = \underbrace{\left(\frac{\partial w}{\partial z}(w_1) \right)^{-1/2}}_{\frac{\beta}{2\pi} \frac{1}{z_1} = \frac{\beta}{2\pi} e^{-\frac{2\pi}{\beta} w_1}} \left(\frac{\partial w}{\partial z}(w_2) \right)^{-1/2} \frac{C}{z_1 - z_2}$$

$$= \frac{2\pi}{\beta} e^{\pi/\beta (w_1 + w_2)} \frac{C}{e^{\frac{2\pi}{\beta} w_1} - e^{2\pi/\beta\, w_2}} = \boxed{\frac{2\pi}{\beta} \frac{C}{2 \sinh\left[\frac{\pi}{\beta} (w_1 - w_2) \right]}}$$

Finite T correlator

spatial correlation: $\langle X_-(x) X_-(0) \rangle \sim 1/x$ for $x \ll \xi_T = \beta/\pi$

$w_1 = x$

$w_2 = 0$

Thermal correlation length

$\sim e^{-x/\xi_T}$ for $x \gg \xi_T$

$\Rightarrow$ finite T cuts off critical (algebraic) correlations.

$L = Na$

id: $k = \frac{2\pi}{L} n = \frac{2\pi n}{Na}$

II RG in a nutshell

lattice $k < \underbrace{\Lambda \sim 1/a}_{\text{maximal momentum}}$ = UV cutoff

Fields $\phi(x) = \sum_{k<\Lambda} e^{ikx} \phi_k$

Main idea: Coarse-grain / integrate out "fast"/high energy/short scale degrees of freedom

$$\phi(x) = \underbrace{\sum_{k<\Lambda/b} e^{ikx}\phi_k}_{\phi_<} + \underbrace{\sum_{\Lambda/b<k<\Lambda} e^{ikx}\phi_k}_{\phi_>}$$

$\phi_>$: momentum shell $b = e^{\ell} \simeq 1+\ell$

- Integrate out $\phi_>$: $Z = \int D\phi\, e^{-S[\phi]} = \int D\phi_< \underbrace{\left(\int D\phi_> e^{-S}\right)}_{e^{-S_{eff}[\phi_<]}}$

easier said than done...

$\Rightarrow$ New effective theory $S_{eff}[\phi_<]$ with cutoff $\Lambda' = \Lambda e^{-\ell}$

- Scale transformation: $k \to kb$ so $\Lambda' = \Lambda/b \to \Lambda$

Now $S_{eff}[\phi_<]$ and $S[\phi]$ have the same cutoff but different couplings: $\{K_i\} \to \{K_i'\} = R_b(\{K_i\})$ RG flow

e.g: Δ in Ising $\Delta \to \Delta'$ etc.

The idea of "the" RG is to iterate this transformation to get rid of microscopic / high energy degrees of freedom, and gain information about coarse grained, long wave-length properties.

- RG fixed points: $\{K_i^*\} = R_\ell(\{K_i^*\})$ Theories invariant under RG transformation

For us, those will be CFT's: conformal invariance.

- Perturbations: Once we've identified a fixed point, we can study its stability against small perturbations $\delta K_i = K_i - K_i^*$.

Couplings that take us away from the fixed point are called Relevant

Perturbations that flow back to K_i^* are called irrelevant.
perturbations that don't flow are called marginal

III Universality of the Ising transition:

- We'll see more explicit applications of the RG later on in this course. For now, we'll adopt a "quick and dirty way" to do RG to leading order, by essentially reducing the problem to dimensional analysis (= "tree level" RG).

Let's go back to the Ising action in the Dirac fermion formulation:

$$S = \int d\tau\, dx \left[\underbrace{\bar{\psi}\, \partial_\tau \psi + \frac{c}{2}\left[\bar{\psi}\partial_x \psi - \psi \partial_x \psi\right]}_{\text{"critical" part of the action} = S_0} + \Delta \bar{\psi}\psi \right]$$

"thermal" perturbation

invariant under $x \to \ell x$, $\psi \to \ell^{-1/2} \psi$
$\tau \to \ell \tau$

$\Delta_\psi = \frac{1}{2}$ Scaling dimension of ψ

. Under this transformation, we have:

$$\Delta \int dx\, d\tau\, \bar{\psi}\psi \;\to\; b\Delta \int dx\, d\tau\, \bar{\psi}\psi$$

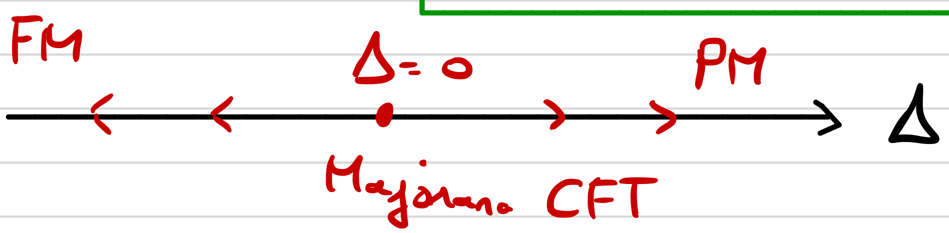
new coupling Δ'

obviously, not invariant.

$\Delta' = b\Delta > \Delta$: coupling grows under RG = relevant perturbation

if we write $b = e^{\ell}$ $\Rightarrow$ $\boxed{\dfrac{d\Delta}{d\ell} = \lim_{\ell \to 0} \dfrac{\Delta' - \Delta}{\ell} = +\Delta}$ $+ \dots$

valid only for small Δ



. Correlation length: Lengthscale $\xi = b\, a$ at which Δ has evolved to be $O(1)$: $\Delta(b) = b\Delta = \xi/a\, \Delta = O(1)$

$$\Rightarrow \boxed{\xi \sim \Delta^{-\nu}} \quad \text{with} \quad \boxed{\nu = 1}$$

. Universality: Say we'd like to study a modified Ising chain:

$$H = -J \sum_i \left(\sigma_i^z \sigma_{i+1}^z + g\, \sigma_i^x\right) + J_1' \sum_i \sigma_i^x \sigma_{i+1}^x + J_2' \sum_i \sigma_i^z \sigma_{i+2}^z + \dots$$

"small" perturbations preserving the Ising symmetry $\mathcal{C} = \prod_i \sigma_i^x$

This model also has a transition between a PM and a FM phase at $g = g_c(J_1', J_2') \neq 1$.

Unlike the Ising model, this model isn't exactly solvable as it maps onto interacting (non quadratic) fermions after a Jordan Wigner transformation (Check this!). However, the universal properties of the transition like the critical exponent $\nu = 1$ are unchanged!
(for all $J_1', J_2', \ldots$)

$$\xi \sim |g - g_c|^{-1} \leftarrow \text{universal}$$
$\neq 1$, non universal

Why? Field theory of the modified model:

$$\mathcal{L} = \bar{\psi}\partial_\tau \psi + \frac{\tilde{c}}{2}(\bar{\psi}\partial_x\bar{\psi} - \psi\partial_x\psi) + \tilde{\Delta}\bar{\psi}\psi + \lambda_1 \bar{\psi}\partial_x^2\bar{\psi} + \lambda_2(\bar{\psi}\partial_x\psi)^2$$

+ ...

also present for $J_1' = J_2' = 0$: higher order term in gradient expansion

interaction

$(\bar{\psi}\psi)^2 = 0$

Now: $\tilde{c} \neq 2J$ and $\tilde{\Delta} \propto \underbrace{(g - g_c(J_1', J_2'))}_{\text{Taylor expand } \Delta = 0 \text{ for } g = g_c} + \ldots$ to leading order

Scale transformation:
$$\lambda_1 \int dx\, d\tau\, \bar{\psi}\partial_x^2\bar{\psi} \longrightarrow \lambda_1/\rho \int dx\, d\tau\, \bar{\psi}\partial_x^2\bar{\psi}$$
$$\lambda_2 \int dx\, d\tau (\bar{\psi}\partial_x\bar{\psi})^2 \longrightarrow \lambda_2/\rho^2 \int dx\, d\tau (\bar{\psi}\partial_x\bar{\psi})^2$$

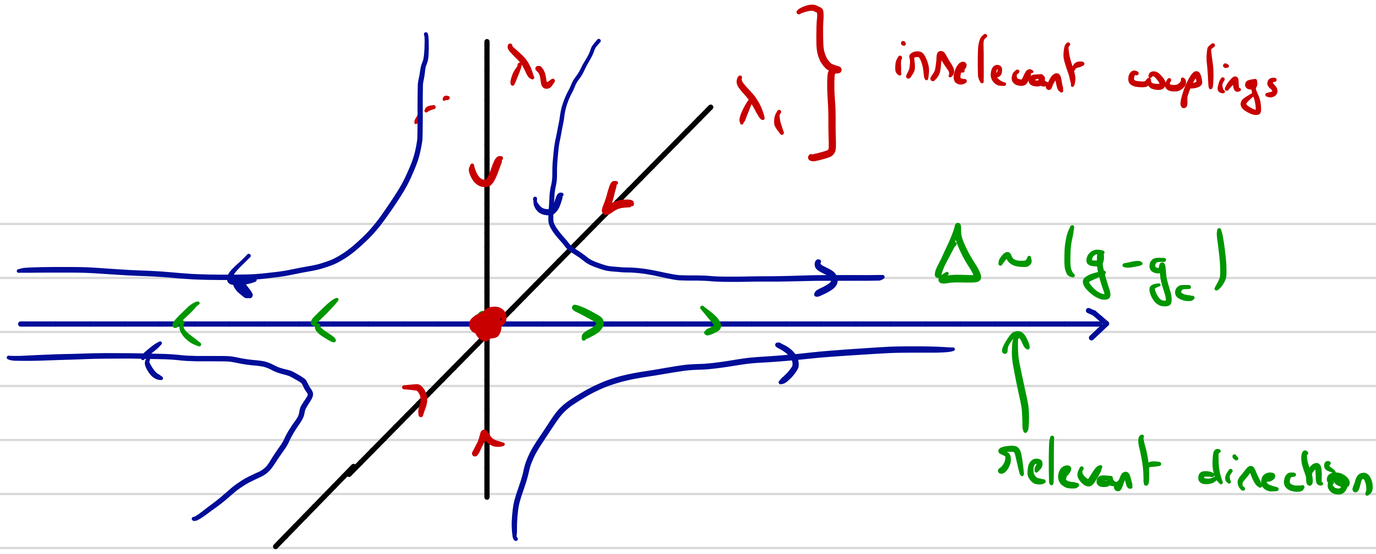
$$\Rightarrow \boxed{\frac{d\lambda_1}{d\rho} = -\lambda_1 \qquad \frac{d\lambda_2}{d\rho} = -2\lambda_2}$$

$\Rightarrow$ irrelevant couplings: decrease under RG / coarse graining.

$\Rightarrow$ can be ignored.

- This justifies a posteriori our gradient expansion: $\psi\partial_x^n\psi$ irrelevant.

- This also explain universality: interactions in Majorana theory $(\bar{\psi}\partial_x\psi)^n$ are irrelevant!



• <u>General leading order RG</u>:

$(D = d+1 = 2$ for our case$)$

$$S_0 = \int d^D x \, \mathcal{L}_0 \quad \text{and} \quad \langle \mathcal{O}(x)\mathcal{O}(0)\rangle \sim \frac{1}{x^{2\Delta_\mathcal{O}}}$$

$\Delta_\mathcal{O}$: scaling dimension of field $\mathcal{O}$

under scale transformation: $\mathcal{O} \longrightarrow b^{-\Delta_\mathcal{O}} \mathcal{O}$

If we now perturb S_0 with $\mathcal{O}$: $S = S_0 + \lambda \int d^D x \, \mathcal{O}(x)$

$$\lambda \int d^D x \, \mathcal{O}(x) \longrightarrow \underbrace{\lambda \, b^{D - \Delta_\mathcal{O}}}_{\lambda'} \int d^D x \, \mathcal{O}(x)$$

λ' = renormalized coupling, $b = e^{\rho}$

$$\Rightarrow \quad \boxed{\frac{d\lambda}{d\rho} = (D - \Delta_\mathcal{O})\lambda} \quad \Longrightarrow \text{relevant if } \Delta_\mathcal{O} < d.$$

<u>Examples</u>: Ising CFT : $\Delta_{\bar{\psi}\psi} = 1$: relevant

$\Delta_{\psi \partial_x^n \psi} = 1 + n$: irrelevant for $n>1$

$\Delta_{(\psi \partial_x^n \psi)^m} = m(1+n)$ dimensional analysis $[\psi] = 1/2$

$[\partial_x] = 1$