Gauge Theories and Topological Order

Gauge Theories and Topological Order

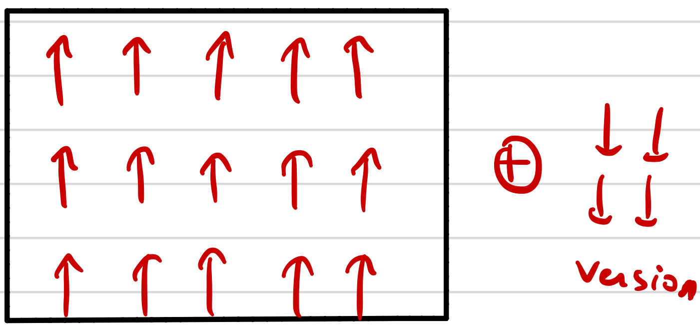
In this chapter: 2+1d systems, phase transition with no local order parameter.

(I) 2+1d Ising Model and Duality

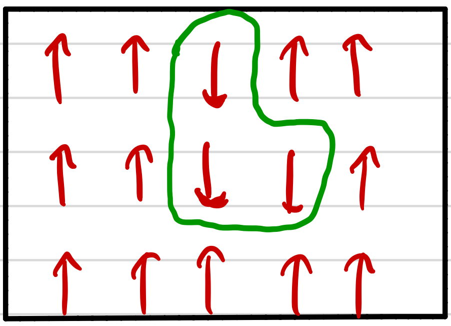
$$H = -J\sum_{\langle ij\rangle}\sigma_i^z\sigma_j^z - h\sum_i\sigma_i^x$$

. For $h \ll J$: FM phase

(almost) degenerate GS



. Excitations:

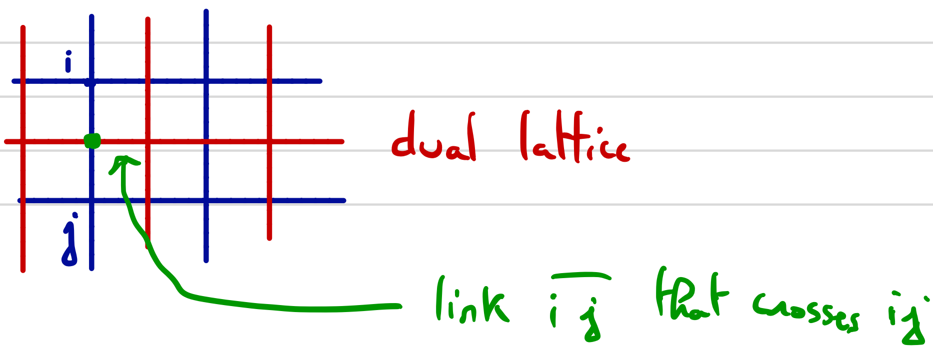


DW: Closed loops, not point-like objects anymore!

. "string like" excitations

. theory of such closed strings = binary version of electromagnetism.

. DW Operator:



Let τ's live on the links of the dual lattice (since DW naturally live on the dual lattice).

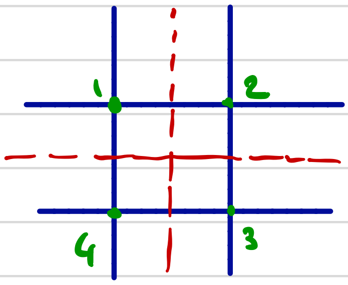
As in 1d:

$$\boxed{\tau^x_{\overline{ij}} = \sigma_i^z \, \sigma_j^z}$$

counts DW between i and j.

<u>However</u>: N sites, $2N$ links $\Rightarrow$ $2N$ τ's!

we need N constraints.



$$\tau^x_{\overline{12}} \, \tau^x_{\overline{13}} \, \tau^x_{\overline{14}} \, \tau^x_{\overline{15}} = (\sigma_1^z \sigma_2^z) \, \sigma_2^z \sigma_3^z \sigma_3^z \sigma_4^z \sigma_4^z \sigma_5^z \sigma_5^z \sigma_1^z$$
$$= 1$$

$\Rightarrow$ local constraint

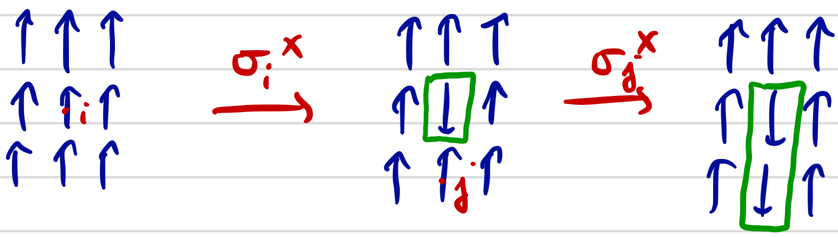
we write:

$$\boxed{\prod_{+} \tau^x_{\overline{ij}} = 1}$$

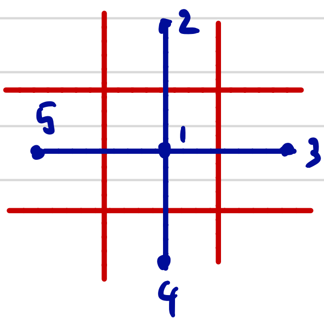
$\forall$ sites of the dual lattice

sets of 4 links that emerge from site i on dual lattice

. <u>Spin flip operator</u>:



$\Rightarrow$ σ_i^x creates loop of DW: flips the value of DW operator τ^x on all bonds surrounding the lattice site i.



$$\sigma_i^x = \tau^z_{\overline{12}} \, \tau^z_{\overline{13}} \, \tau^z_{\overline{14}} \, \tau^z_{\overline{15}} = \prod_{\square_i} \tau^z$$

["plaquette" of the dual lattice]

$$\Rightarrow \boxed{H = -J \sum_{\substack{\text{links} \\ \langle \bar{i}\bar{j} \rangle}} \tau^x_{\bar{i}\bar{j}} - h \sum_{\square} \prod_{\bar{i}\bar{j} \in \square} \tau^z_{\bar{i}\bar{j}}}$$

$\oplus$ Constraints: $\forall \bar{i}$: $\prod_{+} \tau^x = 1$ ← commute: can share 0 or 2 links $(-1)^2 = 1$

Clearly: . the 2+1d Ising model isn't self dual!

. As in 1d, the mapping isn't 1 to 1 (no symmetry breaking and GS degeneracy for the Gauge theory at small h)

II $\mathbb{Z}_2$ Gauge Theory

In the following, we will forget about the Ising model, and treat the dual gauge theory as "fundamental".

$$\boxed{\begin{gathered} H = -J\left[\sum \tau^x + g \sum_{\square} \tau^z\tau^z\tau^z\tau^z\right] \\ \prod_{+} \tau^x = 1 \quad \text{for all } i \end{gathered}}$$

τ's live on links of square lattice

[$\bar{i} \to i$ here]

Gauge invariance: H invariant under local $\mathbb{Z}_2$ gauge transformation:

$$\tau^z_{ij} \to \varepsilon_i \, \tau^z_{ij} \, \varepsilon_j \quad \text{with} \quad \varepsilon_i = \pm 1 \in \mathbb{Z}_2$$

τ^x unchanged.

H is invariant for any such local transformation. This is sometimes called "local symmetry" but this is really a Gauge redundancy of the theory. This transformation doesn't really change states like a spin flip symmetry in the Ising model.

$G_i |\psi\rangle = |\psi\rangle$ states invariant under Gauge transformation

↳ generates Gauge transformation with $\epsilon_i = -1$
$\epsilon_j = 1 \quad \forall j \neq i$

$G_i = \prod_{+_i} \tau^x$ (flips the sign of all τ^z's emanating from i)

$\Rightarrow$ constraints ensures $G_i = 1$: keeps only physical states.

$\mathbb{Z}_2$ Electromagnetism :

$\tau^z_{ij} = e^{i\pi a_{ij}} \qquad a_{ij} = 0, 1$

$$\prod_{\square} \tau^z = e^{i\pi \oint a} \equiv e^{i\pi b}$$ "flux" through plaquette

let $\tau^x = e^{i\pi e}$ so $H \sim (-1)^e + g(-1)^b$

Constraints : $\prod_{+} \tau^x = e^{i\pi \nabla . e} = 1 \Rightarrow \boxed{\nabla . e = 0 \text{ mod } 2}$

where $\sum_{+} e_{ij} = e_{\vec{r}+\hat{y}/2} + e_{\vec{r}-\hat{y}/2} + e_{\vec{r}+\hat{x}/2} + e_{\vec{r}-\hat{x}/2}$

can be flipped to $-$ since $e \equiv -e$

$= \nabla . e$ lattice divergence for $\mathbb{Z}_2$

"$\tau^z_{\bar{i}\bar{j}} = e^{i\pi \int_{\bar{i}}^{\bar{j}} \vec{a} . d\vec{\ell}}$"

Electric Field = DW

Forms closed loops

III Phase diagram of the Ising Gauge Theory

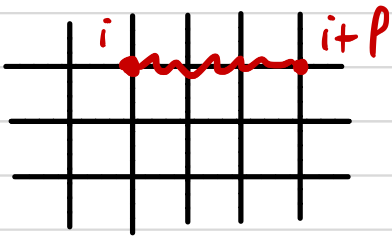
• **Confined Phase:** $g \ll 1$: For $g=0$, $H \simeq -J \sum_{\langle ij\rangle} \tau^x_{ij}$

$\Rightarrow \tau^x = 1$ on all links, satisfies constraint.

This corresponds to $e=0$ everywhere.

. For g small, there will be some links with non-zero electric fields. To satisfy the constraints, the field lines have to form closed loops. For small g, we expect these loops to be small and dilute. As we increase g, these loops proliferate = condense.

• Confinement of test charges: electric lines are confined for $g \ll 1$. To see this, insert two "test charges" at sites i and $i+P$, and ask how much energy it costs to pull these charges apart.



Charges: $\prod_{+_i} \tau^x_{ij} = -1$

(odd number of $\tau^x = -1$ emanate from this site and have to connect to the other test charge)

Each $\tau^x = -1$ links costs energy $2J$: pick shortest path:

$$\boxed{\Delta E(P) = 2J\,P}$$

($\to \infty$ as $P \to \infty$: charges are confined)

• In this phase, the electric field is well defined ($\simeq 0$) while the magnetic field fluctuates wildely.

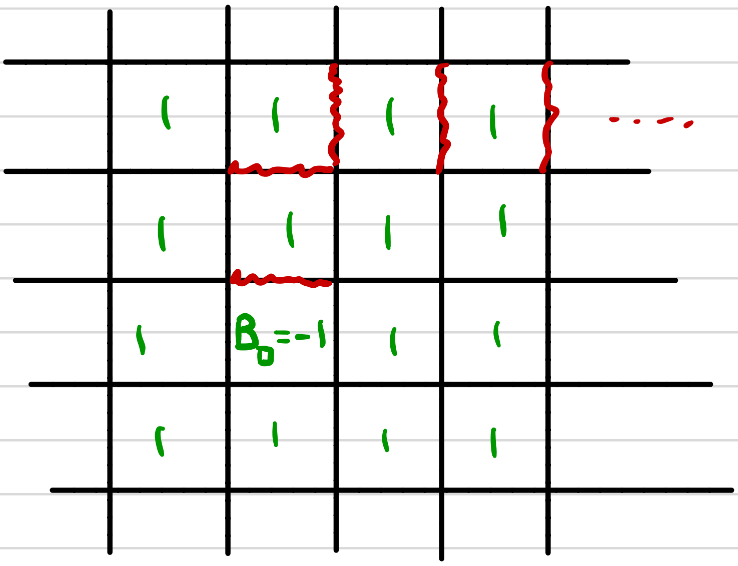
. Remark: it's important to perform this diagnosis in a pure gauge theory (without matter, additional charges)

• Deconfined Phase: $g \gg 1$, $H \simeq -gJ \sum_{\square} \tau^z \tau^z \tau^z \tau^z$

GS: $\prod_{\square} \tau^z_{ij} = +1$ for all $\square$

(Gapped) Excitation: Flip a given plaquette to $\prod_{\square} \tau^z = -1$

energy cost (gap): $\Delta = 2gJ$. To create such an excitation, we actually need to flip flips along a "string"



$B_\square = \prod_\square \tau^z$

- Apply τ^x on ~ links
- This excitation comes with a string attached.

GS wavefunction: Let's first work in the "B" $= \tau^z$ basis.

Naively: $|\psi_0\rangle = \bigotimes_{\langle ij \rangle} |\tau^z_{ij} = +1\rangle$ but not Gauge invariant!

$G_i = \prod_{+} \tau^x$: $\tau^z \to -\tau^z$ on $+$

$\Rightarrow |GS\rangle = \prod_i \underbrace{\frac{(1+G_i)}{2}}_{\text{projection onto } G_i = +1} |\psi_0\rangle$ now $G_i|GS\rangle = G_i$

and $H|GS\rangle = E_0|GS\rangle$ since $[H, G_i] = 0$

$H|\psi_0\rangle = E_0|\psi_0\rangle$

. **Deconfinement of test charges:** insert two test charges $\vec{x}_0$, $\vec{x}_0 + P\hat{x}$

$$|GS'\rangle = \left(\frac{1 - G_{\vec{x}_0}}{2}\right)\left(\frac{1 - G_{\vec{x}_0 + P\hat{x}}}{2}\right) \prod_{j \neq \vec{x}_0, \vec{x}_0 + P\hat{x}} \left(\frac{1 + G_j}{2}\right) |\psi_0\rangle$$

$H|GS'\rangle = E_0 |GS'\rangle \Rightarrow \Delta E(P) = 0$ in this limit!

Charges are deconfined ($\Delta E(P) \underset{P\to\infty}{\not\to} \infty$)

In "E" = τ^x basis: $|\tau^z = +1\rangle = \dfrac{|\tau^x = +1\rangle + |\tau^x = -1\rangle}{2}$

$$= \underset{e=+1}{\bullet\!\!-\!\!-\!\!\bullet} + \underset{e=0}{\bullet \quad \bullet}$$

Up to normalization:
(on simply connected manifold)

$$\boxed{|GS\rangle = \sum_{\substack{\text{loops}\\ \text{configuration } \mathcal{C}}} |\mathcal{C}\rangle}$$

String Condensate

. $\prod_\square \tau^z_{ij} |GS\rangle = |GS\rangle$ as $\prod_\square \tau^z$ creates a loop $\square$

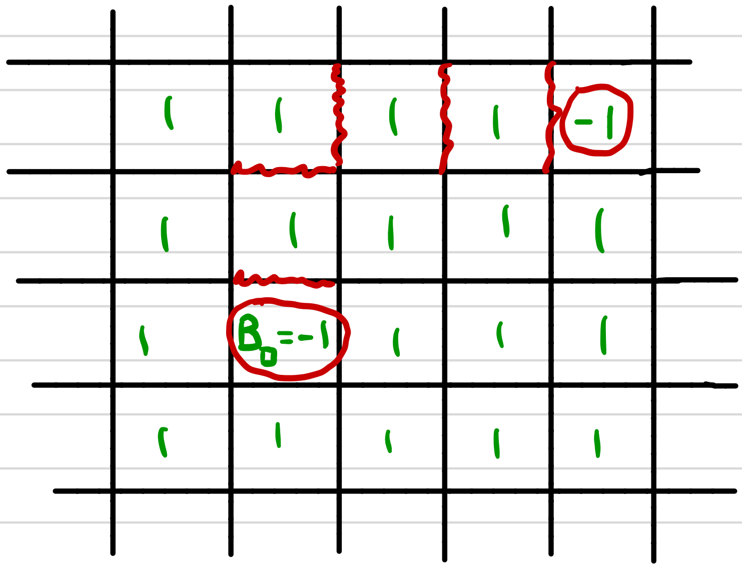
. deconfined phase: strings are "cheap" and fluctuating.

IV Topological Order

Let's consider the deconfined phase.

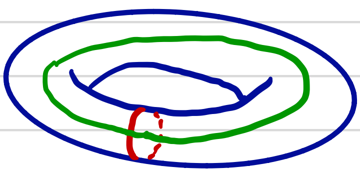
Excitation: gapped magnetic flux excitation = "vison"

carries $\mathbb{Z}_2$ flux of $-1 = \pi$-flux
as explained above, these excitations come with a string

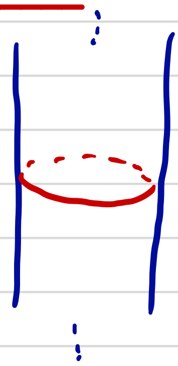


- Acting with τ^x along the string (red links) creates two visons (-1 $\mathbb{Z}_2$ fluxes)
- The string is not measurable by any local measurement. all the plaquettes along the string have no fluxes, $B_\square = +1$.

Topological GS degeneracy: K = # non contractible paths



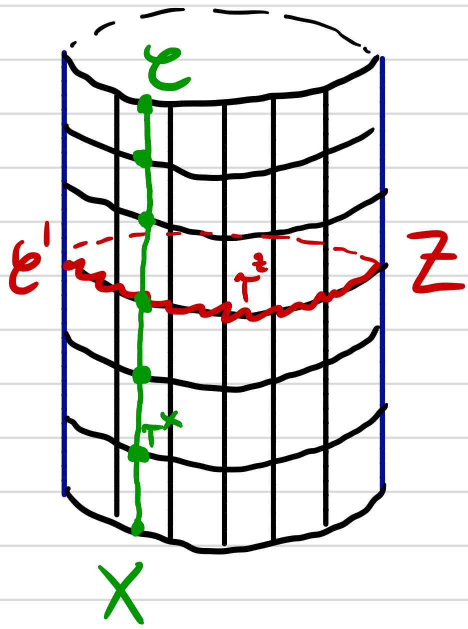
$K = 2$



$K = 1$

GS degeneracy $= 2^K$

Proof: Consider a cylinder and $g = \infty$ limit (for now).



$X = \prod_{e} \tau^x$: creates visons at $\pm\infty$
inserts π flux through the "hole" of the cylinder

$Z = \prod_{e'} \tau^z$: takes electric charge around the cylinder

gauge invariant operators

Take a ground state $|GS\rangle$. Then $X|GS\rangle$ also GS since all plaquettes have 0 $\mathbb{Z}_2$ fluxes through them. $[H,X]=0$

also $[H,Z]=0$.

But $\{Z,X\}=0$ (share one link, and $\{\tau^x,\tau^z\}=0$)

which should be represented on the GS:

$|GS'\rangle = X|GS\rangle$ two-fold degeneracy!

$$Z|GS'\rangle = ZX|GS\rangle = -XZ|GS\rangle = -X\underbrace{Z|GS\rangle}_{+1} = -X|GS\rangle = -|GS'\rangle$$

gauge invariant = physical

$\Rightarrow$ Z measures the $\mathbb{Z}_2$ π flux created by X in $|GS'\rangle$

$\langle GS|GS'\rangle = 0$

More formally, since H commutes with Z and X, the anti-commutation $\{Z,X\}$ should be represented on the GS $\Rightarrow$ implies degeneracy

$Z^2 = X^2 = 1$

<u>Note</u>: Precise contours do not matter. $\mathcal{C}$ can be deformed by acting by G_i.

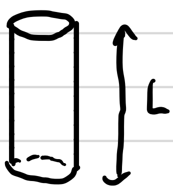
$g \lesssim \infty$: $[H,X]=0$ but $[H,Z]\neq 0$ now.

treat $-J\sum\tau^x$ using perturbation theory.

$|GS\rangle$ and $X|GS\rangle$ now related by matrix element

$H_{eff} = \begin{pmatrix} E_0 & \Gamma \\ \Gamma & E_0 \end{pmatrix}$, actual groundstates are superpositions of $|GS\rangle$ and $X|GS\rangle$

$$\Gamma \simeq J\left(\frac{J}{2Jg}\right)^L$$



- undo string of flipped $\tau^z = -1$.
- at any step, unhappy plaquette energy cost: $2Jg$ (large)

δE = exponentially small in L

⚠ No local operator can tell the difference between $|GS\rangle$ and $|GS'\rangle$, The flux can only be measured by taking an electric charge all around the cylinder.

(V) Toric Code

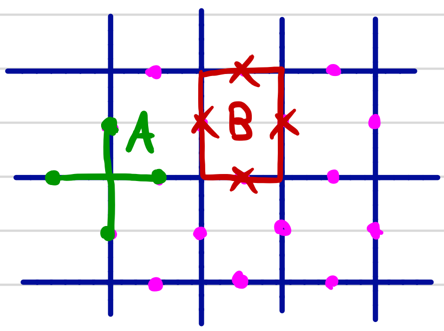
In the $\mathbb{Z}_2$ gauge theory, the physical objects are electric loops (strings), and the Hilbert space doesn't really have a tensor product structure because of the gauge constraint. Can we have this structure emerge in a physical spin model?

$\Rightarrow$ implement constraint "dynamically" (high energy cost for violating it)

$$\boxed{H_{TC} = -J_m \sum_{\square} B_{\square} - J_e \sum_{+} A_{+}}$$

(Kitaev)

with $B_{\square} = \prod_{i\in\square} \tau_i^z$ $\qquad A_+ = \prod_{i\in +} \tau_i^x$



$\square$ = plaquette = p
$+$ = star term = s

no constraint in this model
(but $J_e \to \infty$ enforces previous gauge constraint)

Exact solution: sum of commuting terms: $[A_s, B_p] = 0$
$[A_s, A_{s'}] = 0$
$[B_p, B_{p'}] = 0$
$\forall s, s', p, p'$

GS: $A_s = +1$ $\quad B_p = +1$ $\quad$ ($A_s = +1$ emerges dynamically!)

$B_p = -1$: vortex (magnetic) excitation

$$B_p|GS\rangle = +|GS\rangle \quad \forall p \quad \Rightarrow \quad |GS\rangle = \sum_{\{\tau^z\}} C_{\{\tau^z\}} |\{\tau^z\}\rangle$$

s.t. $\prod_{i\in\square} \tau_i^z = +1 \quad \forall \square$ (no flux)

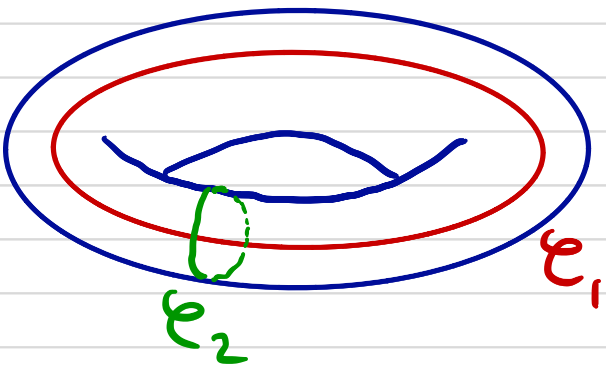
$\Rightarrow$ GS = superposition of vortex-free configurations

Now think of $\tau^z = +1$ as links

$$A_s|GS\rangle = |GS\rangle \quad \forall s$$

. On infinite plane, $C_{\{\tau^z\}} = +1$ up to normalization (as the A_s generate any configuration from $|\{\tau^z_i = +1\}\rangle$)

. On Torus:



$$W_{\mathcal{C}}(\{\tau^z\}) = \prod_{i\in\mathcal{C}} \tau_i^z$$

(sometimes called "Wilson" loops)

Any A_s will intersect 0 or 2 edges of these loops. H_{TC} cannot connect states with different values of $W_{\mathcal{C}_1}$, $W_{\mathcal{C}_2}$

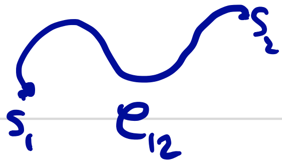
$\Rightarrow$ 4 degenerate GS: $(W_{\mathcal{C}_1}, W_{\mathcal{C}_2}) = (\pm 1, \pm 1)$

Excitations: two flavours: electric charges and magnetic vortices

electric charges $\downarrow$ A term

magnetic vortices $\downarrow$ B term

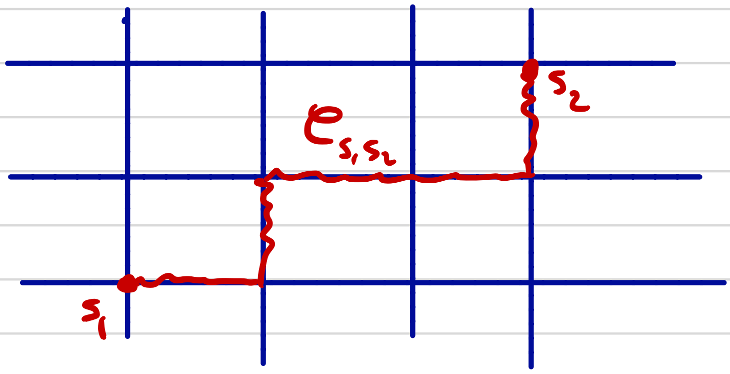
Electric path operator: $$\boxed{W^{(e)}_{e_{s_1 s_2}} = \prod_{i \in e_{s_1 s_2}} \tau_i^z}$$



connect 2 stars

Clearly $[W^{(e)}, B_p] = 0$ $\forall p$, and also commutes with almost all A_s except A_{s_1}, A_{s_2} (share only 1 link)

(anti-commutes)



$|\psi_{s_1 s_2}\rangle \equiv W^{(e)}_{e_{s_1 s_2}} |GS\rangle$

eigenstate with energy $4J_e$.

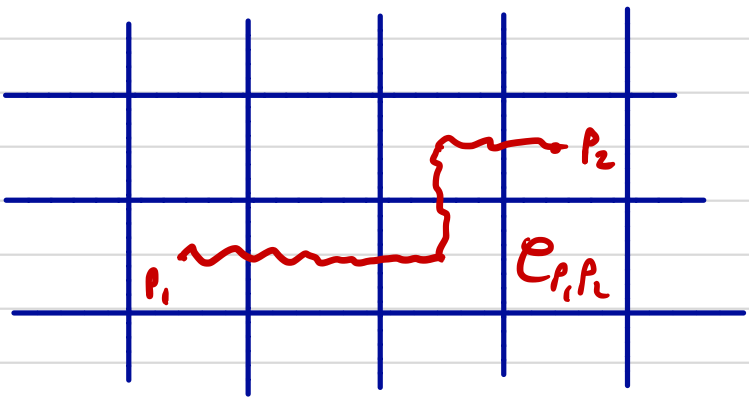
Creates a pair of electric charges at s_1 and s_2

e charge: energy cost $= 2J_e$

Magnetic path operator: $$\boxed{W^{(m)}_{\bar{e}_{p_1 p_2}} = \prod_{i \in \bar{e}_{p_1 p_2}} \tau_i^x}$$

connects two plaquettes,

$\bar{e}$ = path on dual lattice



Commutes with all A_+, and almost all B_p, anti-commutes with B_{p_1} B_{p_2}

$|\psi_{p_1 p_2}\rangle = W^{(m)}_{\bar{e}_{p_1 p_2}} |GS\rangle$

Energy $= 4J_m$

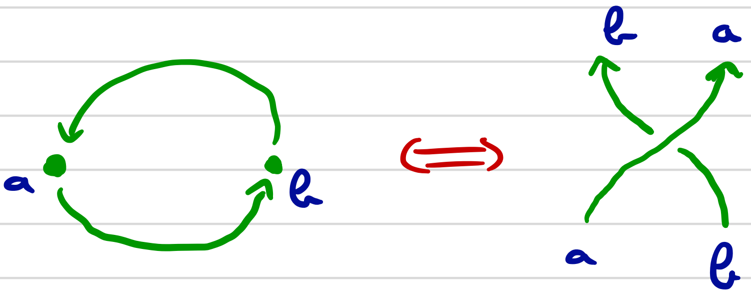
Creates a pair of magnetic vortices at p_1, p_2

m flux: energy cost $= 2J_m$

<u>Note</u>: There's no phase transition in H_{TC} (Commuting projector Hamiltonian)

Anyonic Statistics and Emergent Fermions:

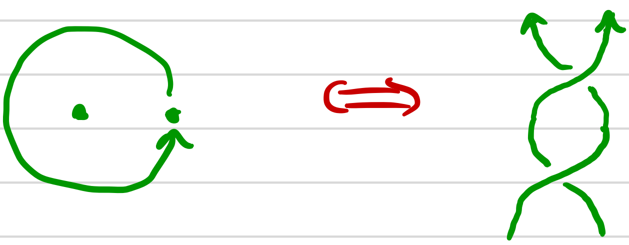
Exchange identical particles, focus on statistical phase:



$= R_{ab}$

indistinguishable particles: $a = b$

Do this twice:



$= R_{ba} R_{ab}$

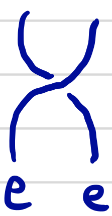
in 3d: Full rotation
two exchanges do nothing $\rightarrow$ identity

in 3d $\longrightarrow$ exchange can lead to ± 1 eigenvalues (fermions in bosons)

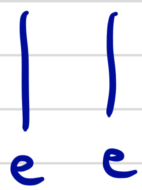
in 2d: Braid group more complicated $\rightarrow$ Anyons.

Toric code: Clearly, e and m are bosons since path operators of the same path commute with each other.

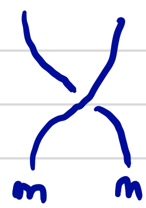
We write:



$=$

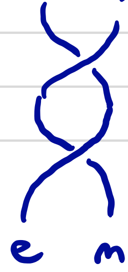


and

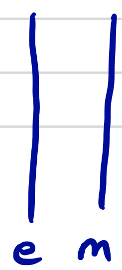


$=$

However, they have some non-trivial mutual statistics:



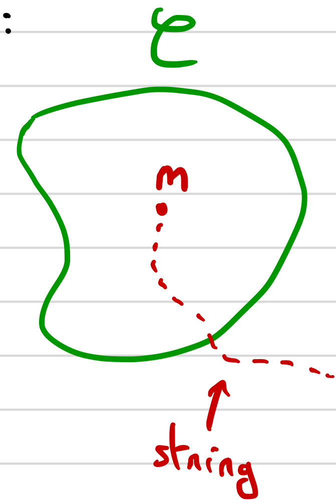
$= -$



Mutual statistics of π

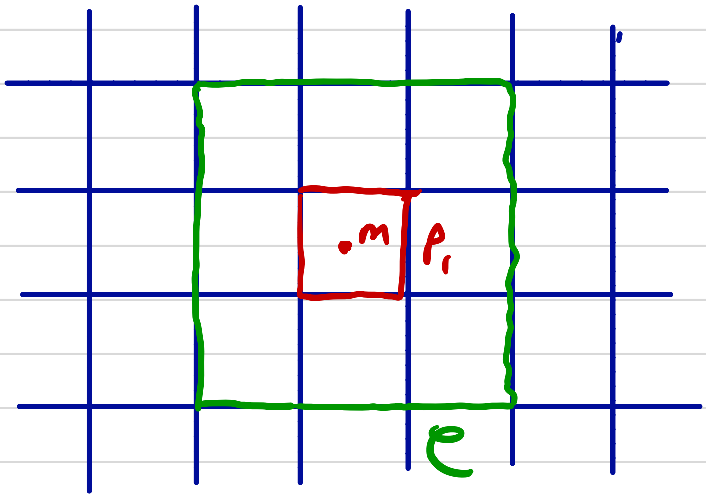
To see this, let's consider $|p_1\rangle$ a state with magnetic vortex at p_1

"Braiding operation": Take e charge around m:



$$|p_1\rangle \longrightarrow \prod_{i\in\mathcal{C}} \tau_i^z \, |p_1\rangle$$

$\mathcal{C}$ ← contour surrounding p_1.



Now:

$$\prod_{i\in\mathcal{C}} \tau_i^z = \prod_{p \text{ inside } \mathcal{C}} B_p$$

("Stokes' theorem!")

$$\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$$

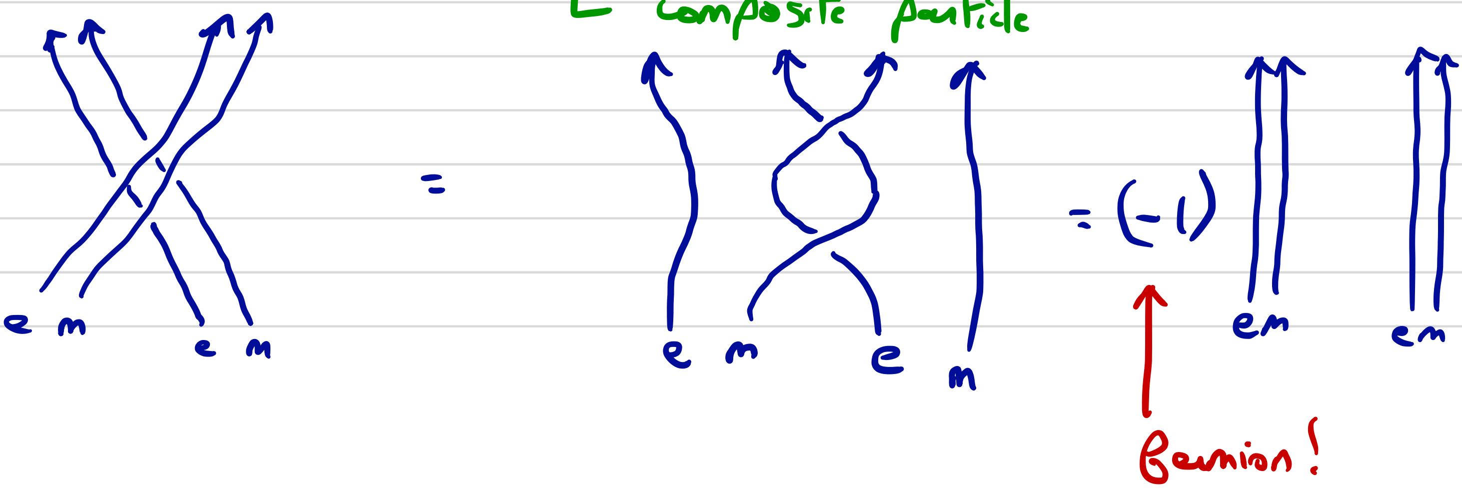
and $B_{p_1}|p_1\rangle = -|p_1\rangle$.

So $|p_1\rangle \longrightarrow -|p_1\rangle$ under this braiding operation.

$\prod_{\mathcal{C}} \tau^z$ isn't trivial if $\mathcal{C}$ encloses a magnetic vortex!

$\Rightarrow$ this means that $\varepsilon = e \times m$ is a fermion:

composite particle



<u>Note</u>: The GS degeneracy can also be understood in terms of this non trivial mutual statistics.

(create ee pair and wrap of of them around a cycle of the torus to annihilate them again, and same thing for mm around the other cycle: those operations anticommute).

(VI) $\mathbb{Z}_2$ Gauge theory with "matter fields"

τ_{ij} : gauge "fields", live on links

σ_i : Ising matter fields, live on sites. (vertex)

$$\boxed{\begin{aligned} H = &- g \sum_{\langle i,j \rangle} \tau_{ij}^x - g^{-1} \sum_{\square} \tau^z \tau^z \tau^z \tau^z \\ &- \lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle i,j \rangle} \sigma_i^z \tau_{ij}^z \sigma_j^z \end{aligned}}$$

} $\mathbb{Z}_2$ pure gauge theory

} Ising Model minimally coupled to σ gauge field

Gauge "symmetry": $\sigma_i^x \prod_{j\, \text{n.n.}\, i} \tau_{ij}^x \quad \Rightarrow \quad \boxed{\prod_{j \in +_i} \tau_{ij}^x = \sigma_i^x}$

$+_i$

$(\nabla . e = \rho)$

$\sigma_i^z \to s_i \sigma_i^z$

$\tau_{ij}^z \to s_i \tau_{ij}^z s_j$ $\quad s = \pm 1$

$\sigma = \mathbb{Z}_2$ electric charge

if $g \to 0$, $\lambda \to 0$ $\quad H = - g^{-1} \sum_{\square} B_{\square} - \lambda^{-1} \sum_{+} A_+$ since $A_{+_i} = \sigma_i^x$

$= H_{TC}$

solved by $\tau_{ij}^z = \hat{\tau}_i^z \hat{\tau}_j^z$
$\hat{\tau}^z = \pm 1$

- $g=0,\ \lambda \neq 0$: pure matter theory: $B_{\square} = +1,\ \forall \square$
(no flux condition)

Under Gauge fixing: $\tau_{ij}^z = +1$ on all links

$$H_{G.F.} = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$

has a symmetry breaking transition as a function of λ. λ large: "Higgs" phase

$\Rightarrow$ conventional symmetry breaking transition upon gauge fixing.

λ give dynamics to e charges. At the Higgs transition, e particles condense (σ^z "gets expectation value")
↳ e creation operator

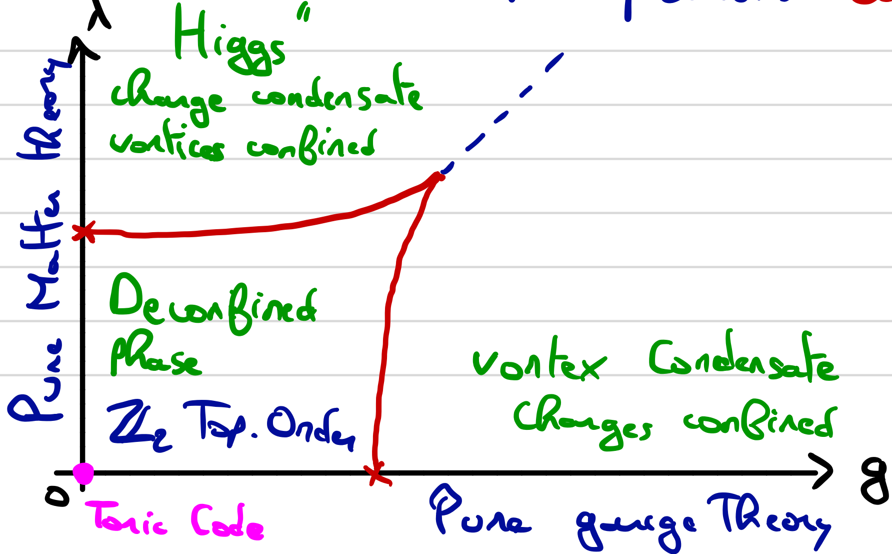
leads to confinement of m (general Topological QFT result: e, m have non trivial mutual statistics)

- $\lambda = 0,\ g \neq 0$: pure gauge theory

electric charges now cost ∞ energy $\Rightarrow$ constraint $\prod \tau^x = +1$ on each star +

$H = H_{\mathbb{Z}_2 \text{ gauge theory}}$.

As g increases: confinement of e particles
m particles condense



Higgs Phase and "Spontaneous Gauge Symmetry Breaking"

- "Xiao-Gang Wen" argument: gauge "symmetries" aren't actual symmetries, "do-nothing" transformation. Two states related by gauge transformation are actually the same state. Can't be spontaneously broken.

- Elitzur's theorem: Gauge symmetries can't be spontaneously broken.

Intuitively: In a 2d classical Ising model, going from all ↑ to all ↓ requires a domain wall with extensive energy cost. In 1d: no extensive energy cost, entropy wins $\Rightarrow$ no FM phase in classical 1d Ising model

→ Same argument breaks down for local gauge symmetries: different GS would be connected by local gauge transformations at no energy cost!

So what's going on in the Higgs phase?

$$H = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle ij \rangle} \sigma_i^z \tau_{ij}^z \sigma_j^z$$

with $B_\square = \tau^z \tau^z \tau^z \tau^z = +1$ (focus on $g \to 0$)

. Gauge fixing: $\tau^z = 1$, then looks like spontaneous symmetry breaking?

. Solution: $\tau_{ij}^z = \sigma_i^z \sigma_j^z$ as $\lambda \to \infty$, $g \to 0$ (satisfies $B_\square = +1, \forall \square$)

The true eigenstates of H can be obtained from:

$$H_{g.f} = -\lambda^{-1} \sum_i \sigma_i^x - \lambda \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z$$

gauge fixed Hamiltonian

by symmetrizing to make them gauge invariant:

$$|\Psi_n\rangle = \sum_G G\,|\Psi_n\rangle_{g.f.} \qquad G = \text{gauge transformations}$$

In particular, the two ferromagnetic groundstates of $H_{g.f.}$

$$|\{\sigma^z=+1,\ \tau^z=+1\}\rangle \quad \text{and} \quad |\{\sigma^z=-1,\ \tau^z=+1\}\rangle$$

are related by gauge transformations.

$\Rightarrow$ the Higgs mechanism looks like SSB for a particular choice of gauge, but the true GS is unique and gauge-invariant

---

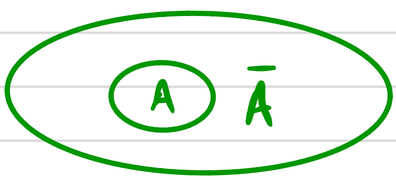
VII Detecting Topological Order using Entanglement

How can one detect deconfinement experimentally or even numerically?

$\rightarrow$ degeneracy on torus with no broken symmetry/order

$\rightarrow$ braiding properties of anyonic excitations.

Say we have the wavefunction $|\Psi\rangle$ that is the G.S. of a topologically ordered H. No torus, no excitation: how do we tell that it is topologically non-trivial?



$\rightarrow$ Entanglement Entropy

$$\boxed{S_A = -\operatorname{tr} \rho_A \log \rho_A}$$

$$\rho_A = \operatorname{tr}_{\bar{A}} \rho \quad \text{with} \quad \rho = |\Psi\rangle\langle\Psi|$$

reduced density matrix

Example: two qubits: $|\Psi\rangle = |\uparrow\uparrow\rangle \Rightarrow \rho = |\uparrow\uparrow\rangle_{AB}\langle\uparrow\uparrow|_{AB}$

$$\rho_A = |\uparrow\rangle_A\langle\uparrow|_A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad S_A = 0$$

not entangled

product state

$$|\psi\rangle = \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \Rightarrow \rho = \frac{1}{2}\Big(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow| + |\downarrow\uparrow\rangle\langle\downarrow\uparrow|\Big)$$

$$\rho_A = \frac{1}{2}\Big(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\Big) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$ Mixed state!

$$S_A = -\sum_i p_i \log p_i = \log 2 \quad \text{with} \quad \sum_i p_i = 1$$

Properties:

- For a pure state: $S_A = S_B$
- Strong subadditivity:

$$S_A + S_B \geq S_{A\cup B} + S_{A\cup B}$$

Many-Body system: $S_A \leq N_A \log 2$ $\quad N_A \sim L_A^2 = \#$ spins $\frac{1}{2}$ in A

for spins $\frac{1}{2}$ in 2d. "volume law"

. Gapped groundstates: $S_A \sim L_A^{d-1}$ area law

. CFT in 1+1d: $S_A = \frac{c}{3}\log L_A$ violates (mildly) area-law

. Low entanglement in quantum GS: tensor networks and matrix product state techniques. DMRG etc.

Entanglement in Toric code: $\boxed{S_A \sim \alpha L_A - \gamma}$

γ: universal, $= \log 2$ for $\mathbb{Z}_2$ top. order

$$|\psi\rangle = \sum_{\substack{\text{electric}\\\text{loops}}} \Big|$$  $$\Big\rangle$$

A

A: compact simply connected.
eigenvalues of ρ_A can be labelled by electric (τ^x) configurations at the boundary.
Loops that do not cross the boundary do not contribute to S_A

Naively 2^{L_A} possibilities for $\tau^x = \pm 1$ at boundary.
all equiprobable in $|\psi\rangle$.

However: Non local constraint from (emergent) Gauss law:
number of electric line crossing boundary is Even.

$\Rightarrow \mathcal{N} = 2^{L_A - 1}$ possible configurations (last e line fixed) Gauss law gives us "one bit" of information.

$p_i = 1/\mathcal{N}$

Non universal, area-law piece

$$S_A = -\sum_{i=1}^{\mathcal{N}} p_i \log p_i = \log \mathcal{N} = L_A \log 2 - \log 2$$

$\gamma = \log 2$

this is a universal property of $\mathbb{Z}_2$ topologically ordered states

---

VIII U(1) Gauge Theories (Very Brief)

Example of lattice U(1) gauge theory in 2+1d:

Compact QED

Consider: 2+1d system, rotors $\theta_p \equiv \theta_p + 2\pi$ defined on links

$$[\theta_p, n_{p'}] = i\,\delta_{p,p'}$$

n_p = conjugate variable = integer

$\theta \in [0, 2\pi)$, n = "angular momentum" if θ = coordinate of particle on a ring

$$e^{-in\hat{\theta}}\,\hat{n}\,e^{in\hat{\theta}} = \hat{n} + m$$

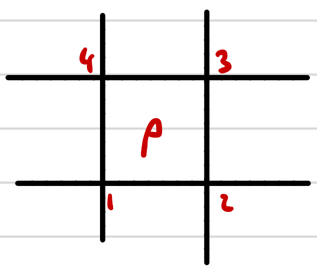
Gauss law: $G_s = \sum_{p \in +} n_p = 0$

- <u>Gauge invariant H:</u> $[H, G_s] = 0$

$H = \alpha \sum_p n_p$ wouldn't have a bounded spectrum from below

$\rightarrow \frac{1}{2} K_E \sum_p n_p^2$: leading electric term.

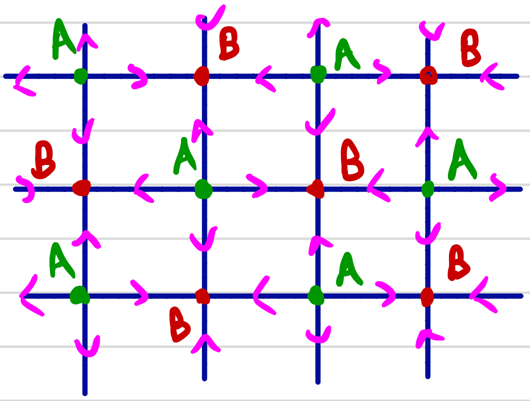
- Since θ is angle, consider operator $e^{i\theta_p} \rightarrow$ not gauge invariant



$$e^{i(\theta_{12} - \theta_{23} + \theta_{34} - \theta_{14})}$$

$\rightarrow$ gauge invariant object.

To get rid of these signs, let's orient the lattice



Draw arrow from $A \rightarrow B$.

$$\boxed{\begin{aligned} e_{ij} &= \varepsilon_i \, n_{ij} \\ a_{ij} &= \varepsilon_i \, \theta_{ij} \end{aligned}}$$

$$\varepsilon_i = \begin{cases} +1 & i \in A \\ -1 & i \in B \end{cases}$$

Gauge constraint : $\underbrace{\nabla . e}_{\in \mathbb{Z}} = 0$ (Gauss law)

$\rightarrow$ charges would be quantized too. Compact QED

Gauge group = $U(1)$

Gauge invariant object : $e^{i \underbrace{(\nabla \times a)}_{b} . \vec{n}_z}$ Magnetic flux through plaquette

$\hookrightarrow = a_{12} + a_{23} + a_{34} + a_{14}$

$$\boxed{H = \frac{K_E}{2} \sum_p e_p^2 - K_B \sum_\square \cos \theta_\square}$$

- $\underline{K_E \gg K_B}$: $e_p \simeq 0$ electric lines costly, confined phase

  $\uparrow$ satisfies constraint

- $\underline{K_B \gg K_E}$: favors $\theta_\square$ small (mod 2π)

  $\longrightarrow \cos \theta_\square \simeq 1 - \theta_\square^2/2$

  $$H \simeq \frac{K_E}{2} \sum_p e_p^2 + \frac{K_B}{2} \sum_\square \theta_\square^2 + \dots$$ $\rightsquigarrow$ Usual QED gapless photons

Similar construction in any dimension.

BUT: in 2+1d, tunneling between minima of $\cos \theta_\square$ crucial!

$\rightarrow$ monopoles (Compact QED)

(Polyakov) $\rightarrow$ confined phase only

$\rightarrow$ photon gets a mass