PHY-897: Special Topics: Solid State Physics, UMass Amherst, Final Exam

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Due: Thursday, May 9

I. SPIN OPERATOR IN THE ISING MODEL

In class, we showed that the critical point of the transverse field Ising chain can be described by a relativistic Majorana field theory, with dynamical exponent $z = 1$ and correlation length exponent $\nu = 1$. However, this fermionic description is not enough to compute all the critical exponents of the Ising model: this is because the magnetization (spin) operator σ^z is *non-local* in terms of Majorana fermions. At the critical point, the spin-spin correlation function decays with an anomalous critical exponent η:

$$G(r) = \langle \sigma_i^z \sigma_{i+r}^z \rangle \sim \frac{1}{r^\eta},$$

where mean-field theory would predict $\eta = 0$ in 1+1d=2D dimension. In this problem, we will use bosonization to compute η.

1. Recall that the critical point $g = 1$ of the quantum Ising chain with N spins can be written as Majorana chain with $2N$ Majorana fermions ξ_j (use a normalization so that $\xi_j^2 = 1$), where each physical site i contains two Majoranas $\xi_{j=2i}$ and $\xi_{j=2i+1}$. Write down the correlator $G(r)$ in terms of these Majorana fermions.
2. Since this correlation function is very hard to compute directly, we will use the following trick: let us introduce two decoupled copies of the Ising model with spins σ and τ (and denote the corresponding Majorana fermions ξ and η). We then write the *square* of the spin correlator $G(r)^2 = \langle \sigma_i^z \sigma_{i+r}^z \tau_i^z \tau_{i+r}^z \rangle$ in this new "doubled" theory. Show that up to "boundary terms" (acting near the sites i and $i + r$), $G(r)^2$ corresponds to the expectation value of the non-local string operator $\mathrm{e}^{\frac{i\pi}{2}\sum_j i\xi_j\eta_j}$ (specify the range of the label j in the sum). In the following, we will admit that the boundary terms do not modify the critical behavior of the correlation function, and can be ignored: $G(r)^2 \sim \langle \mathrm{e}^{\frac{i\pi}{2}\sum_j i\xi_j\eta_j} \rangle$.
3. Form a Dirac fermion ψ out of ξ and η: explain why the Hamiltonian of the doubled theory is a free massless Dirac fermion. Write down the continuum limit of $G(r)^2$ in terms of ψ.
4. Bosonize the fermionic field $\psi(x)$, and write $G(r)^2$ as the two-point function of a bosonic vertex operator. Deduce the value of the critical exponent η.

II. XYZ SPIN CHAIN

The so-called XYZ spin$-\frac{1}{2}$ chain is described by the Hamiltonian

$$H = J \sum_i ((1 + \lambda) S_i^x S_{i+1}^x + (1 - \lambda) S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z).$$

Treat λ as a perturbation of the XXZ spin chain in the gapless regime $-1 < \Delta < 1$, and write the bosonized form of this perturbation. Is this spin chain in a Luttinger liquid state for $\lambda \neq 0$?

III. ANYONS IN THE TORIC CODE

Consider the toric code model in a field, with Hamiltonian

$$H = -J_e \sum_s A_s - J_m \sum_p B_p - \sum_\ell (h_x \sigma_\ell^x + h_z \sigma_\ell^z),$$

where ℓ labels links of the square lattice, and $h \ll J$ so the fields can be treated as perturbations.

1. First consider $h_x = h_z = 0$. In class, we showed that the excitations e and m have non-trivial mutual statistics by taking an e around an m. Obtain the same result, but this time by taking an m particle around an e particle.
2. Show that h_x and h_z induce hopping of the e and m particles. Compute the dispersion relation $\epsilon_{e,m}(k)$ of the particles e and m for non-zero fields. (work in a sector with a single particle e or m.)

IV. IMPURITY IN A LUTTINGER LIQUID

Consider spinless fermions in one-dimension with repulsive interactions, described at low energies by a Luttinger liquid with Luttinger parameter $g < 1$. Now consider a single impurity at position $x = 0$, leading to a term

$$\delta H = V \left(\psi_R^\dagger(x=0)\psi_L(x=0) + \text{h.c.} \right),$$

in the Hamiltonian. This corresponds to an impurity which leads locally to back-scattering at $x = 0$. Using bosonization, write the term in the Euclidian action corresponding to this perturbation, and analyze how it transforms under a scale transformation in Euclidian space. Show that this perturbation is always relevant for repulsive interactions, while it's marginal for non-interacting fermions. This means that for repulsive interactions, the impurity will cut the system in half.