Advanced Statistical Physics: Summary

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The second order phase transition separating paramagnetic (disordered) and ferromagnetic (ordered) phases in the Ising model – and in all uniaxial magnets – can be captured by the effective, coarse-grained, Landau-Ginzburg ϕ^4-theory

$$S = \int d^d x \left(\frac{1}{2} (\nabla \phi)^2 + \frac{t}{a^2} \phi^2 + \frac{u}{a^{4-d}} \phi^4 + \dots \right), \tag{1}$$

where ϕ is the order parameter for the transition (with a $\mathbf{Z}_2$: $\phi \to -\phi$ symmetry), and u, t are dimensionless couplings, with $u > 0$ and t driving the transition. The partition function

$$Z = \int \mathcal{D}\phi \ \mathrm{e}^{-S[\phi]}, \tag{2}$$

can't be evaluated analytically, but a first attempt is to use a saddle-point approximation $\delta S/\delta\phi = 0$. This corresponds to a mean-field approximation that neglects fluctuations, and provides a very simple physical picture of spontaneous symmetry breaking in terms of minimizing the potential $V(\phi) = \frac{t}{a^2}\phi^2 + \frac{u}{a^{4-d}}\phi^4 + \dots$ This predicts a 2nd order phase transition at $t = 0$, with spontaneous symmetry breaking for $t < 0$, with critical exponents $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$. Fluctuations are captured by the two-point function $G(r) = \langle \phi(r)\phi(0) \rangle - \langle \phi \rangle^2$, which within this mean-field approximation satisfies

$$(-\nabla^2 + \xi^{-2}) G = \delta, \tag{3}$$

where $\xi \sim t^{-1/2}$ is the correlation length, which diverges at the transition with the mean-field exponent $\nu = 1/2$. This equation can be solved by Fourier transform, and $G(r)$ decays exponentially as $\sim \mathrm{e}^{-r/\xi}$ off-criticality, while exactly at the critical point $t = 0$, we have

$$G(r) \sim \frac{1}{r^{d-2}}, \tag{4}$$

corresponding to $\eta = 0$.

The saddle-point (mean-field) approximation is ignoring fluctuations. When $d > 4$ (upper critical dimension), the contributions from Gaussian fluctuations about the saddle point to the free energy are subleading (Ginzburg criterion), so that mean-field exponents are exact. On the other hand, if $d < 4$, fluctuations are crucial and invalidate the mean-field predictions. (Note that other universality classes might have different upper-critical dimensions.) If the dimension of the physical system is too small, less than the lower critical dimension, fluctuations can even destroy the long-range ordered phase: for systems with discrete symmetries, the lower-critical is $d = 1$ (in agreement with the absence of transition in the 1d Ising model), while for systems with continuous symmetries, the lower-critical dimension is $d = 2$ (Mermin-Wagner theorem: the Goldstone modes of the putative ordered phase have diverging fluctuations and destroy order).

To deal with these fluctuations more seriously, we use the Renormalization Group (RG). The general idea is to coarse grain by integrating out microscopic degrees of freedom, as in the block spin approach. The partition function is unchanged by this transformation. Schematically, we have

$$Z = \int \mathcal{D}\phi \ \mathrm{e}^{-S[\phi]} = \int \mathcal{D}\phi' \ \mathrm{e}^{-S'[\phi']}, \tag{5}$$

where S' is the effective action for the remaining degrees of freedom ϕ'. During an RG step, the level spacing (UV cutoff) increases $a \to a' = ba$ with $b > 1$. The couplings of the theory "flow" under RG $K_i' = f_i(\{K_j\})$, and fixed points $\{K_j^\star\}$ of these RG equations correspond to stable phases and critical points. The stability of a given fixed point can be analyzed by diagonalizing the matrix $\partial K_i'/\partial K_j$ evaluated at the fixed point. The eigenvalues $\lambda_i = b^{y_i}$ determine whether the corresponding scaling variables $g_i = K_i - K_i^\star$ are relevant ($y_i > 0$, grow under RG), irrelevant ($y_i < 0$), or marginal ($y_i = 0$). The RG framework naturally explains the notion of universality, as different microscopic models can have a transition controlled by the same RG fixed point. It also naturally provides an expression for critical exponents in terms of the RG eigenvalues y_i by exploiting the invariance of the partition function and of the dimensionfull correlation length under RG. For example, the (singular part of the) free energy density obeys

$$f(g_1, g_2, g_3, \dots) = b^{-nd} f(b^{ny_1} g_1, b^{ny_2} g_2, b^{ny_3} g_3, \dots) \tag{6}$$

Let us assume that, say y_1 and y_2 are positive (relevant perturbations, say $g_1 = t/t_0 + \dots$, thermal perturbation being symmetry-even, and $g_2 = h/h_0 + \dots$ field perturbation, odd under symmetry in the case of Ising), while all the other perturbations are irrelevant. We run the RG until $b^{ny_1} g_1 \sim 1$. This yields

$$f(g_1, g_2, g_3, \dots) = g_1^{d/y_1} \Phi(g_1^{-y_2/y_1} g_2, g_1^{|y_3|/y_1} g_3, \dots) \tag{7}$$

Near the transition $g_1 \to 0$, irrelevant variables give vanishing contributions $g_1^{|y_3|/y_1} g_3 \to 0$ and only provide corrections to scaling (assuming this limit is smooth in the function Φ; if that's not the case, those variables are called dangerously irrelevant). Meanwhile, relevant variables give a universal, scaling form for the free energy

$$f(t, h) = \left(\frac{t}{t_0}\right)^{d/y_t} \Phi\left(\frac{h/h_0}{(t/t_0)^{y_h/y_t}}\right). \tag{8}$$

The critical exponents α, β, γ, δ all follow from this scaling form, and can be expressed only in terms of the two relevant eigenvalues y_t and y_h. Applying a similar reasoning to the correlation length, we find $\nu = 1/y_t$. More generally, for any relevant variable g_i, there's a diverging correlation length $\xi \sim g_i^{-1/y_i}$ as $g_i \to 0$ near the transition. This RG framework also naturally incorporates finite size effects: if N is the linear size of the system, N^{-1} is a relevant variable with RG eigenvalue $y_{N^{-1}} = 1$. Relevant couplings g_i correspond to scaling operators (or fields) ϕ_i (not to be confused with the notation ϕ in the ϕ^4 theory!). At criticality, those fields satisfy $\phi_i(r) = b^{-\Delta_i} \phi_i(r/b)$, where $\Delta_i = d - y_i$ is their scaling dimension. Their two point functions read

$$\langle \phi_i(r) \phi_i(0) \rangle \sim \frac{1}{r^{2\Delta_i}}. \tag{9}$$

While block spin implementations of the RG are physically nice and transparent, they are quite cumbersome and uncontrolled in practice. Instead, we consider a real-space perturbative RG expansion near a given fixed point action

$$S = S_0^\star + \sum_i g_i \int \frac{d^d x}{a^{d-\Delta_i}} \phi_i(x), \tag{10}$$

corresponding to a fixed point $S_0^\star$ weakly perturbed by its scaling fields $\phi_i(x)$, with dimensionless couplings $g_i \ll 1$. Within this perturbative regime, we can expand the partition function within perturbation theory, and upon rescaling $a \to ba$ with $b = \mathrm{e}^{\delta\ell} \simeq 1 + \delta\ell$ while keeping the system size unchanged, we find that the couplings have to be modified in the following way to leave the partition function unchanged:

$$\frac{dg_k}{d\ell} \equiv \lim_{\delta\ell \to 0} \frac{g_k' - g_k}{\delta\ell} = y_k g_k - \frac{S_d}{2} \sum_{ij} C_{kij} g_i g_j + \mathcal{O}(g^3), \tag{11}$$

where S_d is the area of the unit sphere in d dimensions. Here, $y_k = d - \Delta_k$ are the RG eigenvalues of the scaling fields ϕ_k, and C_{ijk} are the operator product expansion (OPE) coefficients of those operators, which characterize how the scaling operators can be expanded onto the basis of scaling fields when brought close together. More precisely, we have

$$\phi_i(x) \phi_j(y) \simeq \sum_k \frac{C_{ijk}}{|x - y|^{\Delta_i + \Delta_j - \Delta_k}} \phi_k\left(\frac{x + y}{2}\right) + \dots \tag{12}$$

where this identity holds when inserted into an arbitrary correlation function of the fixed point action $S_0^\star$, as $x \to y$.

For our case of the ϕ^4 theory, the "trivial" fixed point is the Gaussian theory

$$S = \int d^d x \frac{1}{2} (\nabla \phi)^2. \tag{13}$$

The scaling dimension of the field ϕ is $\Delta_\phi = \frac{2-d}{2}$, and follows from naive dimensional analysis (power counting). The scaling dimensions of ϕ^2 and ϕ^4 follow immediately, and to leading order, we have

$$\frac{dt}{d\ell} = 2t + \dots \tag{14}$$

$$\frac{du}{d\ell} = (4-d)u + \dots \tag{15}$$

The Gaussian fixed point corresponds to $u = t = 0$. For $d > 4$, u is irrelevant and $y_t = 2$ ($\nu = 1/2$) drives the transition. This could suggest that the transition in the Ising model for $d > 4$ is simply given by this Gaussian theory, perturbed by $t \int d^d x \phi^2$. However, this is not correct: the scaling dimension of ϕ implies $y_h = \frac{d}{2} + 1$, which leads to critical exponents that coincide with mean-field theory only for $d = 4$. This is because u is dangerously irrelevant for $d > 4$, and it can't be ignored and sent to zero naively. One way to see this is that $u > 0$ is needed to obtained a well-defined ordered phase $\langle \phi \rangle \sim (-t/u)^{1/2}$ within mean-field from $t < 0$.

For $d < 4$, u is relevant so the transition is described by a different fixed point, called Wilson-Fisher (WF) fixed point. We say that the Gaussian fixed point flows to the WF fixed point in the infrared (IR: long wavelengths, low energy) when perturbed by the ϕ^4 term. We can access this fixed point perturbatively in $\epsilon = d - 4$, so that u is barely relevant, and the WF is "close" (in parameter space) to the Gaussian fixed point. We need the OPE structure of the Gaussian to use the equations (11). We have

$$\langle \phi(x) \phi(y) \rangle_0 = \frac{1}{|x-y|^{d-2}}, \tag{16}$$

up to some normalization that we absorb in the definition of the field ϕ. Higher-point correlation functions can be computed using Wick theorem. It is convenient to work with normal ordered operators/fields $:\phi^n:$, which are defined so that Wick theorem applies with no Wick contractions are coinciding points. For example, we have

$$\langle :\phi^2:(x) :\phi^2:(y) \rangle_0 = 2 \langle \phi(x)\phi(y) \rangle_0^2 = \frac{2}{|x-y|^{2(d-2)}}, \tag{17}$$

with $:\phi^2: = \phi^2 - \langle \phi^2 \rangle_0$. Now the OPE structure of those operators follows from Wick theorem:

$$:\phi^2:(x) :\phi^2:(y) \simeq \frac{2}{|x-y|^{2(d-2)}} + \frac{4}{|x-y|^{d-2}} :\phi^2: \left(\frac{x+y}{2}\right) + :\phi^4: \left(\frac{x+y}{2}\right) + \dots \tag{18}$$

This identity is valid when inserted in correlation functions of the Gaussian theory as $x \to y$, with the first term corresponding to Wick contracting all operators in $:\phi^2:(x) :\phi^2:(y)$; the second term corresponds to Wick contracting two of those operators, leaving a term that looks like $:\phi^2:$ from far away. The last term corresponds to all operators in $:\phi^2:(x) :\phi^2:(y)$ being Wick contracted with other operators and not together, so that they look like $:\phi^4:$.