PHY-817: Advanced Statistical Physics, UMass Amherst, Problem Set #3

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Due: Monday, March 23.

I. RENORMALIZATION GROUP TREATMENT OF THE 1D POTTS MODEL

The Q-state Potts model is a generalization of the Ising model where the spins σ_i can take Q values (or "colors"): $\sigma_i = 1, \ldots, Q$. The Hamiltonian reads

$$-\beta \mathcal{H}(\{\sigma_i\}) = K \sum_{<i,j>} \delta_{\sigma_i, \sigma_j},$$

where $\delta_{\sigma_i, \sigma_j}$ is the Kronecker delta function.

1. What is the symmetry group of this statistical model? Show that the $Q = 2$ case can be mapped onto an Ising model.

2. Introduce an order parameter for the Potts model (*Hint: it should have $Q-1$ independent components*).

3. Derive an exact recursion relation for the coupling K in one dimension by performing a real space RG transformation with scale factor $b = 2$. Analyze the fixed points of this recursion relation and their stability. Show that your results are consistent with the absence of an ordered phase for that model in 1D.

II. 2D REAL SPACE RENORMALIZATION GROUP FOR THE ISING MODEL

Consider a 2D Ising model on a triangular lattice in the presence of a magnetic field

$$\beta \mathcal{H}(\{\sigma_i\}) = -K \sum_{<i,j>} \sigma_i \sigma_j - h \sum_i \sigma_i.$$

In class, we carried out a real-space renormalization group procedure in the case $h = 0$ using a cumulant expansion to leading order. Repeat that calculation for $h \neq 0$ and derive an approximate recursion equation for h to leading order. Using this result, compute the critical exponent δ. (Note that the numerical value for y_h quoted in the lecture notes was obtained using a higher-order calculation: your result shouldn't match that value.)

III. FINITE SIZE SCALING AND 2D/3D CROSSOVER IN THE ISING MODEL

(From J. Cardy's book). Consider a three-dimensional Ising model in a slab geometry, with finite thickness L, but effectively infinite in the other two directions. Close to the critical point, if the correlation length ξ is much larger than L, the critical behavior of this model is effectively two-dimensional. In two dimensions, the Ising model is exactly solvable and we know there is a logarithmic singularity in the specific heat at the critical point. For our 3D Ising model in a slab geometry, we thus expect a specific heat singularity of the form

$$c = A(L) \log |T - T_c(L)|,$$

where the parameters $A(L)$ and $T_c(L)$ depend on L. Using the scaling form of the specific heat for the 3D Ising model, derive how the amplitude A should depend on L at large L.

IV. CORRELATION FUNCTIONS

Consider a two-dimensional Ising model at the critical point $T = T_c$ on, say, a square lattice. We are interested in the correlation functions of the lattice "operator" $\mathcal{O}(\vec{r}) = \sigma_{\vec{r}} \sigma_{\vec{r}+\hat{x}} \sigma_{\vec{r}+\hat{y}} \sigma_{\vec{r}+\hat{x}+\hat{y}}$ (product of the four spins on the square/plaquette with lower-left corner at position $\vec{r}$). Argue that its two-point correlation function scales as $\langle \mathcal{O}(\vec{r}) \mathcal{O}(\vec{0}) \rangle - \langle \mathcal{O}(\vec{r}) \rangle \langle \mathcal{O}(\vec{0}) \rangle \sim C r^{-\theta} + \ldots$ where you will express the exponent θ in terms of the standard critical exponents of the Ising model.