PHY-817: Advanced Statistical Physics, UMass Amherst, Problem Set #5

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Due: Friday, May 1.

I. MOMENTUM SHELL RG AND SINE-GORDON THEORY

There are many different ways to implement a renormalization group transformation. In class, we focused on the concept of scaling dimensions and operator product expansions. In many books, you will find another popular approach known as the "momentum shell RG". (It is unfortunately sometimes taken as a definition of "the" RG in some books, whereas it is only one possible RG scheme among many others!) In this problem, we propose to illustrate this approach on the Sine-Gordon theory. The action for the Sine-Gordon model in $d = 2$ dimensions, with UV cutoff $\Lambda \sim 1/a$, is given by

$$S[\varphi] = \int d^2x \left[\frac{1}{2}(\nabla\varphi)^2 + \lambda \cos\beta\varphi \right].$$

Recall that the field $\varphi(x)$ has non-zero Fourier modes only for $k < \Lambda$. The momentum shell RG procedure goes as follows:

1. Decompose the Fourier modes of the field as $\varphi_k = \varphi_k^< + \varphi_k^>$ where $\varphi_k^>$ has support on $\Lambda/b < k < \Lambda$, while $\varphi_k^<$ is non-zero for $0 < k < \Lambda/b$. Let $\varphi^<(x)$ and $\varphi^>(x)$ be the inverse Fourier transform of $\varphi^<(k)$ and $\varphi^>(k)$ respectively. The first step of the RG is to integrate out the "fast" field $\varphi^>$ (with Fourier modes in the small momentum shell $\Lambda/b < k < \Lambda$). The effective action of $\varphi^<$ is defined by
$$\mathrm{e}^{-S'[\varphi^<]} = \int \mathcal{D}\varphi^> \mathrm{e}^{-S[\varphi]}.$$
Show that up to an irrelevant constant and to leading order in λ
$$S'[\varphi^<] = \int d^2x \left[\frac{1}{2}(\nabla\varphi^<)^2 + \lambda\langle\cos\beta(\varphi^< + \varphi^>)\rangle_> \right],$$
where you will define the meaning of $\langle\dots\rangle_>$.
2. Evaluate the expectation value $\langle\cos\beta(\varphi^< + \varphi^>)\rangle_>$.
3. The new theory $S'[\varphi^<]$ has UV cutoff $\Lambda' = \Lambda/b$. In order to compare it to the original theory, the last step of the momentum shell RG is to rescale momentum $k' = bk$ and space $x' = x/b$ to restore the original cutoff $\Lambda/b \to \Lambda$. Find the renormalized coupling λ' of the resulting theory.
4. Conclude that the $\cos\beta\varphi$ potential is relevant when $\beta^2 < 8\pi$, and irrelevant when $\beta^2 > 8\pi$. Recover this result by computing directly the scaling dimension of the operator $\cos\beta\varphi$ by evaluating its two-point function in the Gaussian theory with $\lambda = 0$.

II. CLOCK MODEL

Let us consider the so-called Clock model in two dimensions with Hamiltonian

$$-\beta\mathcal{H} = K \sum_{<i,j>} \cos(\theta_i - \theta_j),$$

where the angles θ_i are restricted to N discrete values $0, \frac{2\pi}{N}, \dots \frac{2\pi(N-1)}{N}$. This model has a discrete $\mathbb{Z}_N$ symmetry, and can therefore have an ordered (symmetry-broken) phase in two dimensions. For $N = 2$, this is simply an Ising model, while for $N \to \infty$ this coincides with the XY model.

1. Argue that this model is in the same universality class as an XY model where the angles are continuous, but with a symmetry-breaking field $-h\sum_i \cos(N\theta_i)$ in $\beta\mathcal{H}$.

2. Let us first ignore vortices by assuming that we can set the vortex fugacity to $y = 0$. In this case, we can ignore the angular nature of θ and replace it with a scalar field ϕ with continuum limit

$$Z = \int \mathcal{D}\phi \mathrm{e}^{-\int d^2x \left(\frac{K}{2}(\nabla\phi)^2 - h\cos N\phi\right)}.$$

   Using the results derived in class for the Sine-Gordon model, write down RG flow equations for K and h.

3. Argue that once vortices are included ($y \neq 0$), the flow equations read

$$\frac{dh}{d\ell} = \left(2 - \frac{N^2}{4\pi K}\right) h,$$
$$\frac{dy}{d\ell} = (2 - \pi K)\, y,$$
$$\frac{dK^{-1}}{d\ell} = -\frac{\pi N^2 h^2 K^{-2}}{4} + 4\pi^3 y^2,$$

   valid for $h, y \ll 1$.

4. Study these flow equations to sketch the phase diagram of this model as a function of temperature. In particular, show that there is an intermediate phase with emergent $U(1)$ symmetry and algebraic quasi-long range order for $N > 4$.

III. SUPERFLUID TRANSITION IN $2 + \epsilon$ DIMENSIONS (NOT GRADED)

Let us consider an XY model with $O(2)$ symmetry in $d = 2 + \epsilon$ expansion.

1. What is the renormalization group equation for the stiffness K of a Gaussian model in $2 + \epsilon$ dimensions?

2. Assuming that the RG equations for the XY model in $2 + \epsilon$ dimensions are simple deformations of those found in class for $d = 2$ (without worrying about what the physical interpretation of y is in this case!), show that there is a non-trivial fixed point describing the finite temperature phase transition in this model.

3. Obtain the eigenvalues at this fixed point to lowest non-trivial order in ϵ, and estimate the exponents ν and α for the superfluid transition in $d = 3$ from these results.