Early Days

P715

The Early Days of Solid State Physics

I Specific heat of solids
= "Heat Capacity" for us

$k_B = 1.38 \times 10^{-23}\ J/K$

Recall: Ideal monoatomic gas: $C_V/N = \frac{3k_B}{2}$

(Equipartition: $k_B/2$ per d.o.f.)

For solids: $C = 3k_B$ per atom ("Law of Dulong - Petit")

at high T. Note: $C_p - C_v = \frac{VT\alpha^2}{k_B T} \simeq 0$ for solids

Model 1: Atoms = harmonic well from interactions

$C = 3k_B$ (3+3 d.o.f: 3 positions, 3 momenta)

$$H_1 = \frac{\vec{p}^{\,2}}{2m} + \frac{k}{2}\vec{x}^{\,2}$$

$$Z = (Z_1)^N \quad \text{and} \quad Z_1 = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \int d^3\vec{x}\, e^{-\beta H_1}$$

$$Z_1 = \frac{1}{(2\pi\hbar)^3}\Bigg[\underbrace{\int_{-\infty}^{+\infty} dp\, e^{-\beta p^2/2m}}_{\sqrt{\pi/(\beta/2m)}} \underbrace{\int_{-\infty}^{+\infty} dx\, e^{-\beta k\frac{x^2}{2}}}_{\sqrt{\pi/\beta k/2}}\Bigg]^3 = (\hbar\omega\beta)^{-3}$$

$\omega = \sqrt{\frac{k}{m}}$

$\langle E \rangle$

$$E = -\frac{\partial \ln Z}{\partial \beta} = 3N k_B T$$ (3N classical harmonic oscillators)

Model 2: Quantum Mechanics for T dependence
(Einstein)

$$Z_1 = \sum_{n_x, n_y, n_z \geq 0} e^{-\beta\hbar\omega\left(n_x+n_y+n_z+3/2\right)}$$

$$= \Big(e^{-\frac{\beta\hbar\omega}{2}} \underbrace{\sum_n e^{-\beta\hbar\omega n}}_{\frac{1}{1-e^{-\beta\hbar\omega}}}\Big)^3 = \left(\frac{1}{2\sinh\left(\frac{\hbar\omega\beta}{2}\right)}\right)^3$$

$$\frac{E}{3N} = -\frac{1}{3N}\frac{\partial \ln Z}{\partial \beta} = \frac{\hbar\omega}{2}\coth\frac{\beta\hbar\omega}{2} = \hbar\omega\left(n_B(\beta\hbar\omega) + \frac{1}{2}\right)$$

with: $n_B(x) = \frac{1}{e^x - 1}$ Bose occupation number

$$c = \frac{\partial (E/N)}{\partial T} = 3k_B(\beta\hbar\omega)^2 \frac{e^{\beta\hbar\omega}}{\left(e^{\beta\hbar\omega}-1\right)^2}$$

High T: $c \underset{\beta\hbar\omega \to 0}{\sim} 3k_B$ (Classical limit)

Low T: "Activated behavior" with gap $\hbar\omega$

$$(\beta\hbar\omega)^2 e^{-\hbar\omega/k_B T}$$

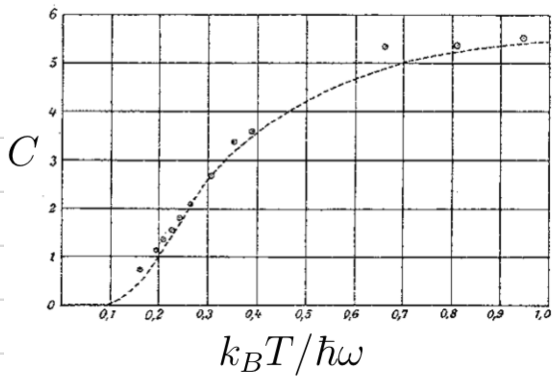


Fig. 2.2 Plot of molar heat capacity of diamond from Einstein's original paper. The fit is to the Einstein theory. The y axis is C in units of cal/(K-mol). In these units, $3R \approx 5.96$. The fitting parameter $T_{Einstein} = \hbar\omega/k_B$ is roughly 1320K. Figure from A. Einstein, *Ann. Phys.*, **22**, 180, (1907), Copyright Wiley-VCH Verlag GmbH & Co. KGaA. Reproduced with permission.

$$T_{\text{Einstein}} = \frac{\hbar\omega}{k_B}$$

usually $\ll T_{\text{room}}$ though there are exceptions.

Pretty good qualitatively but not at low T.

Debye ~1912

Model 3: $T \ll T_{\text{Einstein}}$, $C^{exp} \sim T^3$

Not exponential / activated. Indicates "gapless" modes

vibration modes in solid ~ sound waves (phonons)

$$\omega = v|\vec{k}|$$

Recall waves in a box:

$$e^{i\vec{k}\cdot\vec{x}} \qquad x = L \equiv 0$$

$$\Rightarrow \quad \vec{k} = \frac{2\pi}{L}\vec{n}, \qquad \vec{n} \in \mathbb{Z}^3$$

If L large enough:

$$\sum_{\vec{k}} = \sum_{\vec{n}} = \int d^3\vec{n} = \frac{L^3}{(2\pi)^3}\int d^3\vec{k}$$

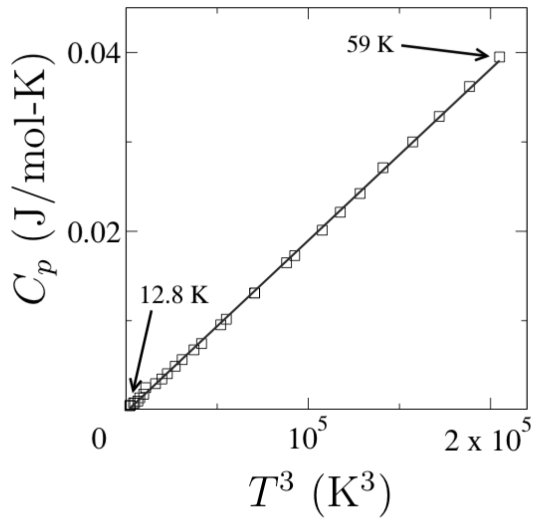


Fig. 2.3 Heat capacity of diamond is proportional to T^3 at low temperature. Note that the temperatures shown in this plot are far far below the Einstein temperature and therefore correspond to the very bottom left corner of Fig. 2.2. Data from Desnoyehs and Morrison, *Phil. Mag.*, **3**, 42 (1958).

polarizations, 2 transverse, 1 longitudinal

$$E = 3 \sum_{\vec{k}} \hbar\omega(\vec{k}) \left[n_B\left(\beta\hbar\omega(\vec{k})\right) + \frac{1}{2} \right]$$

$$= \frac{3L^3}{(2\pi)^3} \int d^3k \; \hbar\omega(\vec{k}) \left[n_B\left(\beta\hbar\omega(\vec{k})\right) + \frac{1}{2} \right]$$

with $\int d^3k = \int_0^{k_D} dk \; 4\pi k^2 (\ldots)$

since $\omega(\vec{k})$ depends only on $k = |\vec{k}|$

k_D = UV cutoff. $\vec{k} \in$ Brillouin zone (see later)

$k_D^3 L^3 \sim 3N \Rightarrow k_D \sim \left(\frac{N}{V}\right)^{1/3} \sim \frac{1}{a}$ (smallest k set by lattice spacing)

and $$E = \frac{3L^3}{2\pi^2} \int_0^{k_D} dk \; k^2 \hbar\omega(k) \left[\frac{1}{e^{\beta\hbar\omega(\vec{k})} - 1} + \frac{1}{2} \right]$$

$\frac{1}{2} \to E_0$ T-independent

$$E = E_0 + \frac{3L^3}{2\pi^2} \int_0^{k_D} dk \; \frac{\hbar v k^3}{e^{\beta\hbar v k} - 1}$$

$x = \beta\hbar v k$

$$= E_0 + \frac{3L^3}{2\pi^2} \frac{(k_BT)^4}{(\hbar v)^3} \int_0^{\beta\hbar v k_D = T_D/T} dx \; \frac{x^3}{e^x - 1}$$

$T \gg T_D$

High T: $E = E_0 + 3\sum_{\vec{k}} k_B T$

$\simeq E_0 + 3N k_B T$

with $3\sum_{\vec{k}} = \#\text{ states} = 3N = \frac{3L^3}{2\pi^2}\int_0^{k_D} dk\, k^2 = \frac{(k_D L)^3}{2\pi^2}$

$T_D \sim 10^2 K$

$\Rightarrow k_D = (6\pi^2 n)^{1/3} \sim$ inverse lattice spacing.

Low T: $E = E_0 + \frac{3L^3 k_B^4 T^4}{2\pi^2 \hbar^3 v^3} \underbrace{\int_0^\infty dx \frac{x^3}{e^x - 1}}_{\pi^4/15}$

$E \sim T^4 \quad \Rightarrow \quad C \sim T^3$ ✓

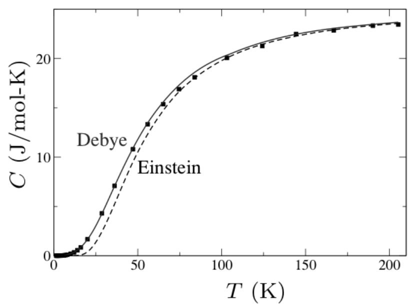


Fig. 2.4 Heat capacity of silver compared to the Debye and Einstein models. The high-temperature asymptote is given by $C = 3R = 24.945$ J/(mol-K). Over the entire experimental range, the fit to the Debye theory is excellent. At low T it correctly recovers the T^3 dependence, and at high T it converges to the law of Dulong–Petit. The Einstein theory clearly is incorrect at very low temperatures. The Debye temperature is roughly 215 K, whereas the Einstein temperature roughly 151 K. Data is taken from C. Kittel, *Solid State Physics*, 2ed Wiley (1956).

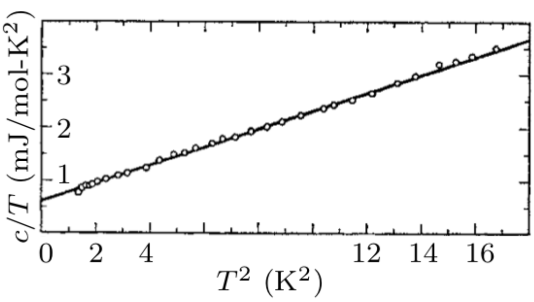
Very good agreement.

We'll fix some of the remaining issues later:

- $\omega \sim v k$ only at low k.
- Details of the crystal $k \in BZ$
- $C = \alpha T + \beta T^3$

from electrons → next section

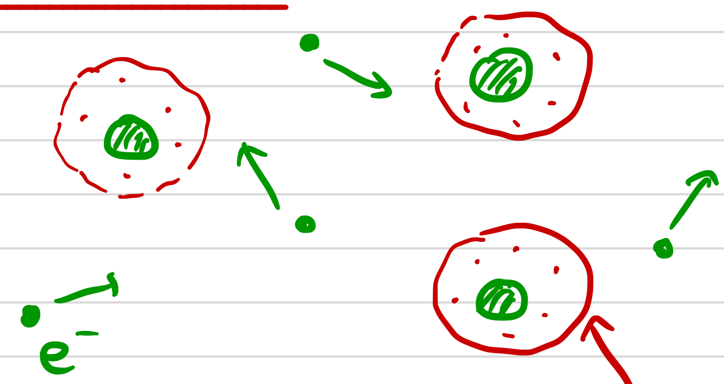
phonons



II Electrons in Metals

Metals: Excellent conductors (heat and electricity), shiny, malleable, ...

Drude Model



valence (free) e^-

nuclei + core e^-, fixed

Gas density: $n = \dfrac{N_A Z \rho_m}{A}$

\# valence e^-

molar density (mol/m³)

Avogadro's number: 6.02 × 10²³

atomic mass number (# protons + neutrons)

Volume per e^-: $\dfrac{1}{n} = \dfrac{4}{3}\pi r_s^3$ → $r_s \sim 2$ to 3 Bohr radius (1.2 Å)

Density: ~100 that of a typical gas.

Assumptions:

- Collisions with rate $1/\tau$ (τ: mean free time)

  τ = phenomenological parameter here

- Between collisions, e^- are free and independent.

- Collisions randomize velocity $\langle \vec{p} \rangle = \vec{0}$ after collision
- ignore $e^- - e^-$ interactions, and think of e^- scattering off impurities in a classical way.

$$\langle \vec{p}(t+dt) \rangle = \underbrace{\left(1 - dt/\tau\right)}_{\text{no collision: classical motion}} \left(\vec{p}(t) + \vec{F}(t)\, dt\right) + \vec{0}\; \underbrace{dt/\tau}_{\text{prob. to scatter}}$$

$$\Rightarrow \quad \frac{d\langle \vec{p} \rangle}{dt} = -\langle \vec{p} \rangle/\tau + \vec{F}$$

$\vec{F}$: external force ($\vec{E}$, $\vec{B}$ fields)

drop: $\langle \vec{p} \rangle \to \vec{p}$.

$$\vec{p}(t) = \vec{p}(0)\, e^{-t/\tau}$$

Momentum relaxation by scattering

DC response: $\vec{E} \neq \vec{0}$, $\vec{B} = \vec{0}$

$\vec{F} = -e\vec{E}$. In steady state: $\vec{p} = -e\tau\vec{E}$

electric current: $\vec{j} = -en\vec{v} = \frac{ne^2\tau}{m}\vec{E}$

$$\Rightarrow \quad \boxed{\sigma = \frac{ne^2\tau}{m}}$$

Drude conductivity

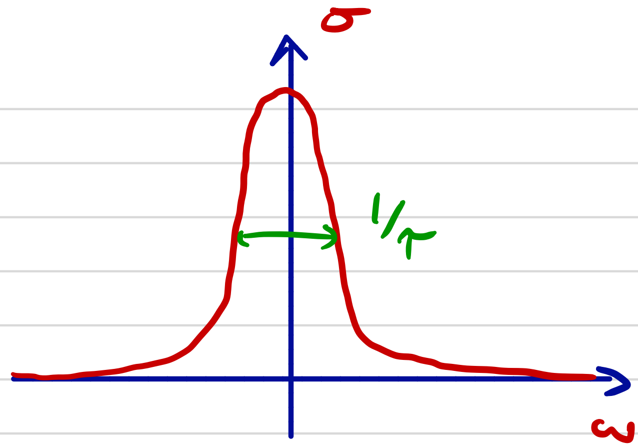
AC response: $\vec{E} = \vec{E}_0 e^{-i\omega t}$ $\quad \vec{p} = \vec{p}_0 e^{-i\omega t}$

$$-i\omega \vec{p}_0 = -\vec{p}_0/\tau - e\vec{E}_0$$

$$\Rightarrow \quad \vec{p}_0 = m\vec{v}_0 = \frac{-e\vec{E}_0}{-i\omega + \tau^{-1}}, \quad \vec{j} = \frac{ne^2\tau/m}{1 - i\omega\tau}\vec{E}$$

$$\boxed{\mathrm{Re}\,\sigma(\omega) = \frac{ne^2\tau/m}{1+\omega^2\tau^2}}$$

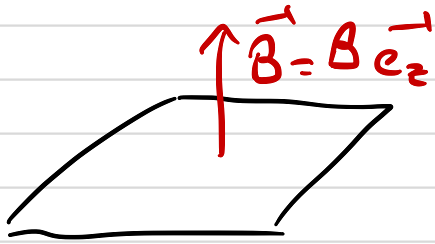
Lorentzian



Note: as $\tau \to \infty$: $\mathrm{Re}\,\sigma(\omega) = \pi \dfrac{ne^2}{m} \delta(\omega)$
(no collision)

Drude weight

since $\delta(\omega) = \lim_{\varepsilon \to 0} \dfrac{1}{\pi} \dfrac{\varepsilon}{\omega^2 + \varepsilon^2}$



Hall response: $\vec{E} \neq \vec{0}$, $\vec{B} \neq \vec{0}$

$$\frac{d\vec{p}}{dt} = \vec{0} = -e\left(\vec{E} + \vec{v} \times \vec{B}\right) - \vec{p}/\tau$$

and $\vec{j} = -ne\vec{v} = -\dfrac{ne}{m}\vec{p}$

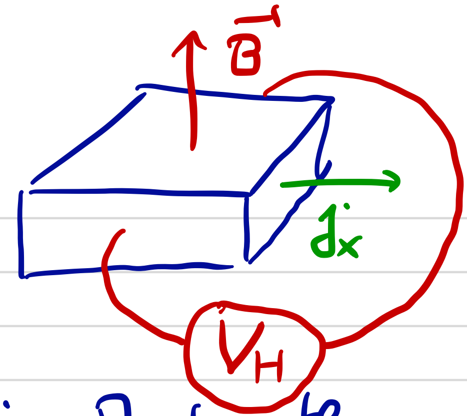
$$\Rightarrow \quad \vec{E} = \left(\frac{1}{ne}\vec{j} \times \vec{B} + \frac{m}{ne^2\tau}\vec{j}\right)$$

$$\vec{E} = \frac{1}{ne}\begin{vmatrix} j_x \\ j_y \\ j_z \end{vmatrix} \times \begin{vmatrix} 0 \\ 0 \\ B \end{vmatrix} + \frac{m}{ne^2\tau}\begin{vmatrix} j_x \\ j_y \\ j_z \end{vmatrix} = \begin{vmatrix} j_y B/ne + \frac{m}{ne^2\tau} j_x \\ -j_x B/ne + \frac{m}{ne^2\tau} j_y \\ 0 + \frac{m}{ne^2\tau} j_z \end{vmatrix}$$

$$= \hat{\rho}\,\vec{j} \qquad \text{with} \qquad \hat{\rho} = \begin{pmatrix} m/ne^2\tau & B/ne & 0 \\ -B/ne & m/ne^2\tau & 0 \\ 0 & 0 & m/ne^2\tau \end{pmatrix}$$

resistivity matrix

Off diagonal term = Hall resistivity



Hall coefficient : $\boxed{R_H = \dfrac{\rho_{yx}}{|B|} = -\,{}^{1}\!/_{ne}}$ in Drude theory

→ can measure sign and density of charge carriers

Has to be negative in Drude's theory. Comes out with opposite sign in some materials

Hall effect + σ_{DC} → estimate $\tau \sim 10^{-14}$ s for most metals at room temperature.

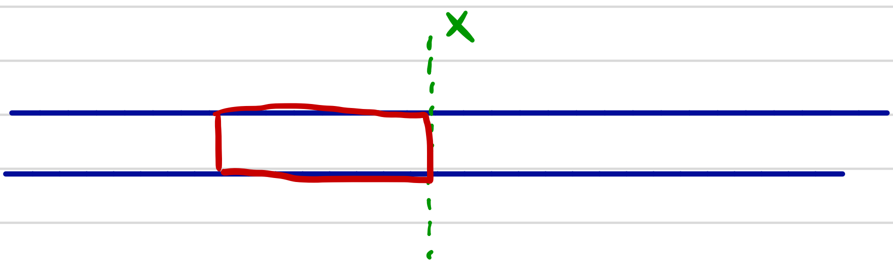
Thermal transport : neglect contribution from ions

(metals conduct heat better than insulators)

$$\vec{j}^{\,Q} = -\kappa \vec{\nabla} T \qquad \text{(Fourier's law)}$$

energy current — thermal conductivity

1d model



$$j^Q = \frac{1}{2} n v \left[\underbrace{\mathcal{E}(x - v_x\tau)} - \mathcal{E}(x + v_x\tau) \right] = -v_x^2 \tau n\, \partial_x \mathcal{E}$$

energy of e^- coming from the left
energy from last collision

$$\underbrace{\partial_T \mathcal{E}}_{c}\, \partial_x T$$

$$\kappa = c_v n \tau \underbrace{\langle v_x^2 \rangle}_{\frac{1}{3}\langle \vec{v}^2 \rangle}$$

by isotropy

Assume : ideal gas

$$C_v = \frac{3}{2} k_B$$

$$\frac{3}{2} k_B T = \frac{1}{2} m \langle \vec{v}^2 \rangle$$

(Maxwell-Boltzmann distribution)

$$\Rightarrow \kappa = \frac{3}{2} \frac{n \tau k_B^2 T}{m} \quad \text{and} \quad \sigma = \frac{n e^2 \tau}{m}$$

Wiedemann-Franz Law

$$\Rightarrow \boxed{\frac{\kappa}{\sigma T} = \frac{3}{2} \left(\frac{k_B}{e}\right)^2} \approx 1.11 \times 10^{-8} \ W \Omega . K^{-3}$$

Lorenz number

- Known to be nearly universal accross all metals "triumph"
- Off by factor of 2 :

However : $C \neq \frac{3}{2} k_B$ in metals, much smaller and linear in T

$\langle v^2 \rangle$ is much larger

Two errors of 10^2 that cancelled out each other!

Thermoelectric response : Peltier effect : e^- current carries heat

$$\vec{j}^{\dot{Q}} = \Pi \vec{j}$$

In kinetic theory : $\vec{j}^{\dot{Q}} = \frac{1}{3} C_v T \, n \vec{v}$ ← heat carried by particle

$C_v = \frac{3}{2} k_B$

$$\vec{j} = - e n \vec{v}$$

Peltier coefficient: $\Pi = -\frac{c_v T}{3e} = -\frac{k_B T}{2e}$

Thermopower = Seebeck coefficient $= \Pi / T = -k_B/2e$

$\simeq -4.3 \times 10^{-4}$ V/K

Actual value $\sim 10^2$ times smaller!!

(III) Sommerfeld Theory

Electrons obey Fermi-Dirac statistics.

Replace in Drude theory: $dP = f(\vec{v})\, \underbrace{d^3\vec{v}}_{4\pi v^2 dv}$

$$f(\vec{v}) = \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-mv^2/2k_BT}$$

Maxwell-Boltzmann

$$f(\vec{v}) = \frac{(m/\hbar)^3}{4\pi^3}\, n \, \frac{1}{e^{(\frac{1}{2}mv^2 - k_B T_F)/k_B T} + 1}$$

Fermi-Dirac

Reminders from P602 on Fermi Gases:

(B.I) Fermi-Dirac distribution

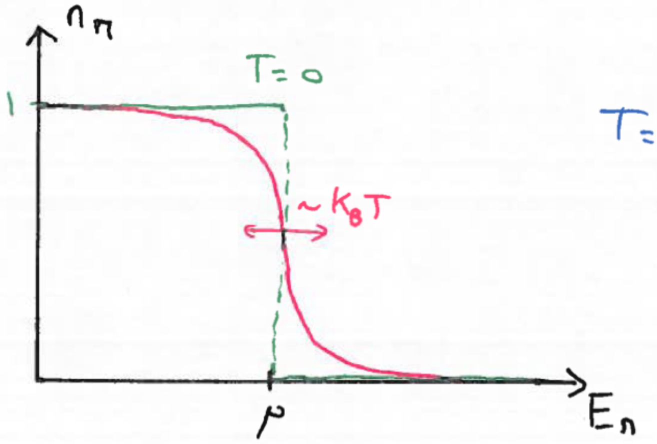
Non-interacting fermions: e^- in metals (Landau theory), ^3He, white dwarfs, neutron stars, ...
Fermi liquids

spin $s =$ half integer : Pauli exclusion: $\psi(\vec{r}_1, \vec{r}_2) = -\psi(\vec{r}_2, \vec{r}_1)$

$$Z = \prod_n \sum_{\boxed{n_n = 0,1}} e^{-\beta n_n (E_n - \mu)} = \prod_n \underbrace{\left(1 + e^{-\beta(E_n - \mu)}\right)}_{Z_n}$$

$$\langle n_n \rangle = \frac{1}{\beta} \frac{\partial}{\partial \mu} \ln Z_n \implies \boxed{n_n = \frac{1}{1 + e^{\beta(E_n - \mu)}}}$$

Fermi-Dirac Distribution
(Note: $\mu \in \mathbb{R}$ here!)



$T = 0$: Fill states with $E_n < E_F \equiv \mu(T=0)$ ← at fixed N

↑ Fermi energy

Def: $\boxed{T_F = E_F / k_B}$ ($\sim 10^4$ K in metals)

Fermi Temperature

Recall: $E_k = \frac{\hbar^2 k^2}{2m}$ $\qquad dE = \frac{\hbar}{m} k\, dk$

$$2 \sum_k = 2V \int \frac{d^3k}{(2\pi)^3} = \frac{2V}{2\pi^2} \int k^2 dk = 2 \int \frac{V}{2\pi^2} \sqrt{\frac{2Em}{\hbar^2}} \frac{m}{\hbar} dE$$

↑ spin

$$= \int g(E)\, dE \quad \text{with} \quad g(E) = \frac{V}{2\pi^2} \sqrt{E} \left(\frac{2m}{\hbar^2}\right)^{3/2}$$

Density of states

and $\lambda_T = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$ (thermal de Broglie length)

B.II Ideal Fermi Gas at high temperature

$$E = \frac{\hbar^2 \vec{k}^2}{2m} \quad \text{and} \quad \sum_n = \sum_{\vec{k}} = \int g(E)\,dE \quad \text{with} \quad g(E) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} g_s \sqrt{E}$$

$g_s = 2s+1$ spin degeneracy

$g_s = 2$ for e^-

Very similar to bosons: $N = \int_0^\infty dE \frac{g(E)}{z^{-1}e^{\beta E}+1}$, $E = \int_0^\infty dE \frac{E\, g(E)}{z^{-1}e^{\beta E}+1}$

$$pV = -\Omega = \frac{1}{\beta}\ln \mathcal{Z} = \frac{1}{\beta}\int_0^\infty g(E)\ln\left(1+e^{-\beta E}z\right) = \frac{2}{3}E \qquad z = e^{\beta\mu}$$

integration by parts

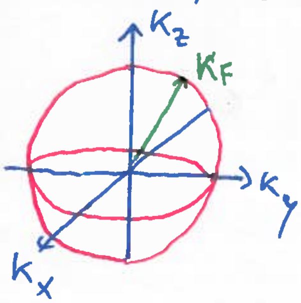
$z \ll 1$: $\frac{1}{z^{-1}e^{\beta E_n}+1} \simeq z e^{-\beta E_n} \longrightarrow$ Boltzmann distribution

Small $z \ll 1$ expansion $\Longrightarrow$ (cf HW and boson case)
$$\boxed{pV = N k_B T\left(1 + \frac{\lambda^3 N}{4\sqrt{2}\, g_s V} + \dots\right)}$$

FD statistics increases p at high T: Pauli exclusion!

B.III Degenerate Fermi Gas at low T

At $T=0$, fill states with $E = \frac{\hbar^2 k^2}{2m} \leqslant E_F = \frac{\hbar^2 k_F^2}{2m}$: $\boxed{|\vec{k}| < k_F}$



Fermi sea with Fermi surface $|\vec{k}| = k_F$

What is E_F? $T=0 \Rightarrow \frac{1}{z^{-1}e^{\beta E}+1} = \Theta(E-\mu)$ with $\mu = E_F$

$$N = \int_0^{E_F} g(E)\,dE = \frac{g_s V}{6\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2} \Longrightarrow \boxed{E_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g_s}\frac{N}{V}\right)^{2/3}}$$

Energy: $E = \int_0^{E_F} E\, g(E)\,dE = \frac{3}{5} N E_F \Longrightarrow \boxed{pV = \frac{2}{5} N E_F}$

Degeneracy pressure at $T=0$ from Pauli exclusion!

important for neutron stars and white dwarf stars

Note: we can either use $g(E)$, with $N = \int_0^{E_F} g(E)\,dE$ and $E = \int_0^{E_F} E\,g(E)\,dE$

or work in K space: $N = g_s \frac{V}{(2\pi)^3} \int_{|\vec{k}|<k_F} d^3\vec{k} = g_s \frac{V}{(2\pi)^3} \frac{4}{3}\pi k_F^3 \Rightarrow k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$ with $g_s = 2$

and $E = g_s V \int_{|\vec{k}|<k_F} \frac{\hbar^2 k^2}{2m} \frac{d^3\vec{k}}{(2\pi)^3} = \frac{g_s V \overbrace{\hbar^2}^{= E_F/k_F^2}}{2m\,(2\pi)^3} \cancel{4\pi} \frac{k_F^5}{5} = \frac{3}{5} N E_F$

At low but finite T: expect particles within $\Delta E \simeq k_B T$ of E_F to be excited

$\Rightarrow$ $\Delta E\, g(E_F) \simeq k_B T\, g(E_F)$ particles carrying energy $k_B T$

$$\frac{E}{V} = \frac{3}{5}\underbrace{\left(\frac{N}{V}\right)}_{n = \text{density}} E_F + \text{cst}\; g(E_F)(k_B T)^2$$

with $N \sim E_F^{3/2} \sim E_F\, g(E_F) \ldots \sim T_F\, g(E_F)$

$$\Rightarrow \boxed{C_V \simeq N k_B \frac{T}{T_F}}$$

(up to prefactors, see below!)

B.IV Sommerfeld expansion and the specific heat of metals

More rigorous treatment: $N = \int_0^\infty \frac{g(E)}{e^{\beta \varepsilon} z^{-1} + 1}\,dE = \underbrace{\frac{V g_s}{(4\pi^2)} \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{1}{\beta^{3/2}}}_{= \frac{2}{\sqrt{\pi}} g_s \frac{1}{\lambda_T^3} V} \int_0^\infty \frac{\sqrt{x}\,dx}{e^x z^{-1} + 1}$

$x = \beta E$

$$\Rightarrow \boxed{\frac{N}{V} = \frac{g_s}{\lambda_T^3} f_{3/2}(z)} \quad \text{with} \quad \boxed{f_n(z) = \int_0^\infty \frac{x^{n-1}}{z^{-1} e^x + 1} \frac{dx}{\Gamma(n)}}$$

(again, very similar to boson calculation)

Similarly: $$\boxed{\frac{E}{V} = \frac{3}{2} \frac{k_B T}{\lambda_T^3} g_s f_{5/2}(z)}$$

At low T, we need to expand these integrals using the fact that the denominator is nearly a step function $\longrightarrow$ Sommerfeld expansion.

Consider integrals of the form: $K = \int_0^\infty \frac{\phi(x)\,dx}{z^{-1}e^x + 1}$ with $\phi(0) = 0$, $\phi(x)e^{-x} \xrightarrow[x\to\infty]{} 0$

$z = e^{\beta\mu} \to \infty$: denominator nearly a step function, its derivative is nearly $\delta(x-\beta\mu)$

integrate by parts: $K = -\int_0^\infty F(x) \underbrace{\frac{d}{dx}\left(\frac{1}{z^{-1}e^x+1}\right)}_{\text{very peaked around } x \sim \beta\mu}$ with $F(x) = \int_0^x \phi(x)\,dx$, $\frac{dF}{dx} = \phi(x)$

$\Rightarrow$ expand $F(x) = F(\beta\mu) + (x-\beta\mu)\underbrace{F'(\beta\mu)}_{\phi(\beta\mu)} + \frac{(x-\beta\mu)^2}{2}\underbrace{F''(\beta\mu)}_{\phi'(\beta\mu)} + \ldots$

we find: $K = F(\beta\mu)\int_{-\beta\mu}^\infty \frac{d}{dy}\left(\frac{-1}{e^y+1}\right)dy + \phi(\beta\mu)\int_{-\beta\mu}^\infty y\,\underbrace{\frac{d}{dy}\left(-\frac{1}{e^y+1}\right)}_{\frac{e^y}{(1+e^y)^2} = \frac{1}{(e^{y/2}+e^{-y/2})^2}}dy$

$y = x - \beta\mu$

$-\beta\mu \to -\infty$; odd, even $\Rightarrow 0$

$+ \frac{\phi'(\beta\mu)}{2}\underbrace{\int_{-\infty}^{+\infty} y^2 \frac{1}{4\cosh^2\frac{y}{2}}dy}_{\pi^2/3} + \ldots$

$$\Rightarrow \boxed{K = F(\beta\mu) + \frac{\pi^2}{6}\phi'(\beta\mu) + \ldots}$$

$F(\beta\mu) \to \int_0^{\beta\mu}\phi(x)\,dx$: step function contribution

Using this result: $\frac{N}{V} = \frac{\cancel{g_s}}{\lambda_T^3}\frac{2}{\sqrt{\pi}}\left[\frac{2(\beta\mu)^{3/2}}{3} + \frac{\pi^2}{6}\frac{1}{2\sqrt{\beta\mu}} + \ldots\right] = \frac{\cancel{g_s}}{6\pi^2}\left(\frac{2m}{\cancel{\hbar^2}}\right)^{3/2}E_F^{3/2}$ ← def. of E_F

$\frac{g_s}{\lambda_T^3}$: $g_s\left(\frac{m k_B T}{2\pi\cancel{\hbar^2}}\right)^{3/2}$

$$\Rightarrow E_F^{3/2} = \mu^{3/2}\left[1 + \frac{\pi^2}{8}\frac{1}{(\beta\mu)^2} + \ldots\right]$$

this yields $E_F = \mu\left[1+\frac{\pi^2}{12}\frac{1}{(\beta\mu)^2}+\dots\right] \Longrightarrow \boxed{\mu = E_F\left[1-\frac{\pi^2}{12}\underbrace{\left(\frac{T}{T_F}\right)^2}_{\frac{1}{\beta E_F}}+\dots\right]}$

and $$\frac{E}{V} = \frac{3}{2}\frac{k_B T}{\lambda_T^3}\, g_s\, \overbrace{\frac{4}{3\sqrt{\pi}}}^{1/\Gamma(5/2)}\left[\frac{2}{5}(\beta\mu)^{5/2}+\frac{\pi^2}{6}\,\frac{3}{2}(\beta\mu)^{1/2}+\dots\right]$$

$$= \frac{3}{5}\frac{k_B T}{\lambda_T^3}\, g_s\, \frac{4}{3\sqrt{\pi}}(\beta\mu)^{5/2}\left[1+\frac{5\pi^2}{8}\underbrace{\frac{1}{(\beta\mu)^2}}_{\simeq\frac{1}{(\beta E_F)^2}=\left(\frac{T}{T_F}\right)^2}+\dots\right] = \frac{3}{5}\underbrace{\frac{N E_F}{V}}_{T=0\ \text{result}}\left[1+\pi^2\underbrace{\left[\frac{5}{8}-\frac{5}{24}\right]}_{\frac{5}{12}}\left(\frac{T}{T_F}\right)^2+\dots\right]$$

$(\beta\mu)^{5/2} \to E_F^{5/2}\left[1-\frac{5\pi^2}{24}\left(T/T_F\right)^2\right]$

$\Longrightarrow C_V = \frac{N E_F}{T_F}\frac{\pi^2}{2}\left(\frac{T}{T_F}\right) \Longrightarrow \boxed{c_V = \frac{\pi^2}{2}k_B\frac{T}{T_F}}$

per particle

To get c per unit volume multiply by n

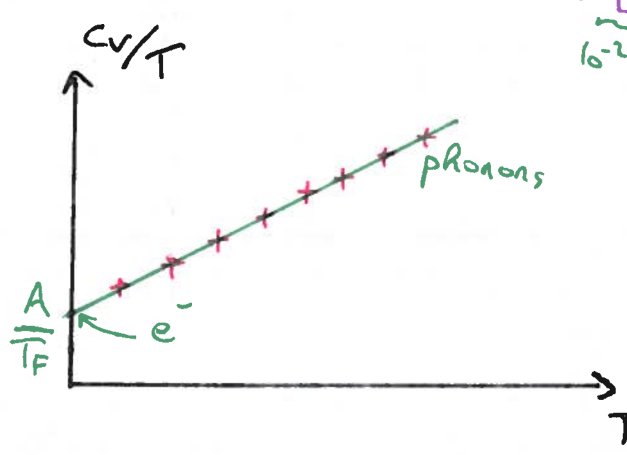
Heat capacity of metals: at low T — phonons $\sim T^3$; $e^- \sim T$ (ideal gas? $\Rightarrow$ Fermi Liquid)

$$\boxed{C_V = A\frac{T}{T_F}+B\left(\frac{T}{T_D}\right)^3+\dots}$$

typically: $T_D \sim 10^2\,K$

$T_F \sim 10^4\,K$

comparable for $T^* = \left(\frac{A\,T_D^3}{\underset{10^{-2}}{B}\ \underset{10^{2}}{T_F}}\right)^{1/2} \simeq O(1K)$



Matches experiments very well

We have: $2\frac{V}{(2\pi)^3}d^3\vec{k}$ levels, occupied with proba n_k

We have $N = 2\sum_k n_k = 2\frac{V}{(2\pi)^3}\int d^3\vec{k}\, n_k$

$\vec{v} = \frac{\hbar}{m}\vec{k}$

$$= \frac{2V}{(2\pi)^3}\int d^3\vec{v}\left(\frac{m}{\hbar}\right)^3 n_{\vec{k}=\frac{m\vec{v}}{\hbar}}$$

gives velocity distribution. Note that $T \ll T_F$ always since $T_F \sim 10^4 K$.

$$\Rightarrow \boxed{\begin{array}{l} C_v = \frac{\pi^2}{2} k_B\, T/T_F \\ v \simeq v_F \end{array}}$$

T-dependent, about 10^{-2} classical result from T/T_F

$\frac{1}{2}mv_F^2 = k_B T_F$

v_F^2 T independent, very large (can be $10^{-2} \times c$!)

<u>Thermal conductivity</u>:

$$\kappa = \frac{1}{3}nv^2\tau c_v = \frac{1}{3}n \overset{2E_F/m}{v_F^2}\,\tau\,\frac{k_B^2 T}{E_F}\frac{\pi^2}{2}$$

$$= \frac{\pi^2}{3}\frac{n\tau}{m}k_B^2 T$$

$$\Rightarrow \boxed{\frac{\kappa}{\sigma T} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 \simeq 2.44 \times 10^{-8}\, W.\Omega.K^{-2}}$$

Thermopower: $\frac{\Pi}{T} = -\frac{c_v}{3e} = -\frac{\pi^2}{6}\frac{k_B}{e}\frac{T}{T_F}$

smaller than Drude by a factor $T/T_F \sim 10^{-2}$ at room temperature.

Note: Mean free path $\lambda \sim v_F \tau \sim 10^2$ Å

- Why don't e^- scatter off ions? ($\rightarrow$ Bloch theorem)
- Core e^- don't contribute
- Charge of charge carriers? (cf. Hall coeff. "holes")
- Role of $e^- e^-$ interactions? (Fermi Liquid theory)