PHY-715: Solid State Physics, UMass Amherst, Final Exam

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Due: Friday May 7.

I. CHERN INSULATOR AND CHIRAL EDGE STATES

In class we saw that a band with Chern number $C = 1$ filled with spinless electrons leads to a quantized Hall conductivity $\sigma_{xy} = \frac{e^2}{h}$ even in the absence of magnetic field (quantum anomalous Hall effect). In this problem, we propose to establish that this Hall conductivity is arising from *gapless* (metallic) chiral edge states, while the bulk of the system is insulating. For this purpose, we consider the Haldane model:

$$\mathcal{H} = -i\hbar v_F(\tau^z\sigma^x\partial_x + \sigma^y\partial_y) + m_{\mathrm{AB}}\sigma^z + m_{\mathrm{H}}\sigma^z\tau^z.$$

Here the first term is the low-energy Dirac Hamiltonian of graphene, written in real space, where the pseudo-spin σ refers to the A, B sublattice space, and $\tau^z = \pm 1$ labels the $\mathbf{K}, \mathbf{K}'$ points (two valleys). The term $m_{\mathrm{AB}}\sigma^z$ is a "mass term" that breaks the inversion symmetry between A and B sites (in this problem, we assume $m_{\mathrm{AB}} > 0$), and $m_{\mathrm{H}}\sigma^z\tau^z$ is the Haldane mass term that breaks time-reversal symmetry.

1. Assuming that the system is in a trivial phase for $m_{\mathrm{AB}} \gg m_{\mathrm{H}}$, determine the phase diagram of this model as a function of m_{H}. You should find 3 distinct phases with Hall conductivity $\sigma_{xy} = 0, \pm\frac{e^2}{h}$. (Recall that the Chern number of a Dirac Hamiltonian $\mathcal{H}_D = \hbar v_F(\pm k_x\sigma^x + k_y\sigma^y) + m\sigma^z$ is $C = \pm\frac{\mathrm{sign}(m)}{2}$.)

2. Consider now an interface between a Chern insulating phase with $C = \pm 1$ for $y < 0$, and a trivial insulator in the upper half-plane $y > 0$. At low-energies, this situation can be described by a single Dirac Hamiltonian $\mathcal{H} = -i\hbar v_F(\mp\sigma^x\partial_x + \sigma^y\partial_y) + m(y)\sigma^z$ with a mass term $m(y)\sigma^z$ changing sign at $y = 0$ and $m(y) > 0$ for $y > 0$. Show that this Hamiltonian admits eigenstates of the form $\psi_{k_x}(x, y) = \mathrm{e}^{ik_x x}\phi(y)$ with energy $E = \pm\hbar v_F k_x$, where $\phi(y)$ is a function localized near $y = 0$ (exponential decaying as a function of $|y|$). Those solutions correspond to gapless, chiral edge modes that are either right- or left-moving depending on the Chern number.

II. 2D TOPOLOGICAL INSULATOR AND QUANTUM SPIN HALL EFFECT

In this problem, we propose to study the so-called "2d topological insulator" state, also known as the quantum spin Hall insulator, which a 2d state of matter that has a quantized spin-Hall conductivity and a vanishing charge Hall conductivity. Here we follow the first proposal for the existence of a quantum spin Hall state that was developed by Charles Kane and Gene Mele in 2005, building on the Haldane model. The main idea is to notice that taking into account the spin degree of freedom of electrons, one can write a new mass term for the low-energy graphene Hamiltonian:

$$\mathcal{H} = -i\hbar v_F(\tau^z\sigma^x\partial_x + \sigma^y\partial_y) + m_{\mathrm{SO}}\sigma^z\tau^z s^z,$$

where $s^z = \pm 1$ labels the physical spin $\uparrow, \downarrow$ of the electrons. This last term does not break any symmetry (in particular it is time-reversal invariant), and should thus be present in graphene. Physically, it arises because of *spin orbit-coupling.*

1. Using the results of the previous problem, show that the total Hall conductivity of this system is $\sigma_{\mathrm{xy}} = 0$. (*Hint:* Notice that this model can be mapped onto two copies of the Haldane model, one for each spin species.)

2. Consider instead the spin Hall current $j_s = \frac{\hbar}{2e}(j_\uparrow - j_\downarrow)$ (say along $\vec{e}_x$) as the response to a transverse electric field $\vec{E} = E\vec{e}_y$, where $j_{\uparrow,\downarrow}$ correspond to the charge (electric) current due to electrons with spin $\uparrow, \downarrow$ so the total electric current is $j = j_\uparrow + j_\downarrow$. Compute the spin-Hall conductivity σ^s_{xy} associated with j_s in this system.

3. Using the previous problem, describe the nature of the edge states of this 2d topological insulator phase.

In practice, graphene has extremely weak spin-orbit coupling, and observing the quantum spin Hall state would require very low temperatures. However, this two-dimensional topological insulator state was also predicted by Bernevig, Hughes and Zhang to occur in quantum wells (very thin layers) of mercury telluride sandwiched between cadmium telluride in 2006, and was observed experimentally in 2007 by König et al.