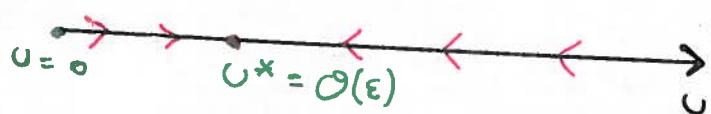


The perturbative renormalization group

Idea: small parameter $\varepsilon \ll 1$ such that non trivial fixed point is close to a "trivial" (say Gaussian) fixed point when ε is small. \Rightarrow controlled calculation

Single coupling example



u is marginal
for $\varepsilon = 0$, and relevant

$$\frac{du}{d\mu} = -Ku(u - u^*) + \dots$$

$$= Ku^*u - Ku^2 + \dots$$

say $Ku^* > 0$: $u = 0$ unstable
 $u^* = O(\varepsilon)$: perturbative stable
fixed point

Scaling operator at $u = 0$ fixed point with coupling λ :

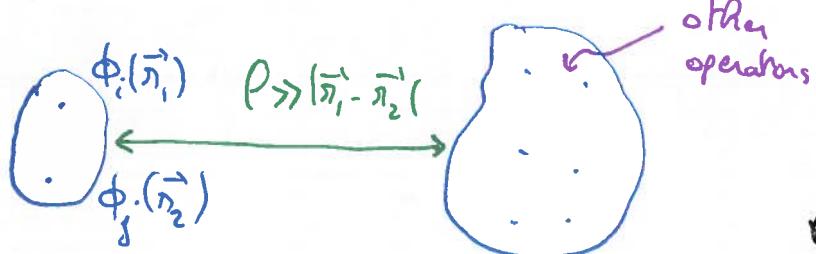
$$\frac{d\lambda}{d\mu} = \gamma_\lambda \lambda + \gamma_u \lambda + \dots$$

$$\gamma_\lambda \rightarrow \gamma_\lambda + \underbrace{\gamma_u}_{{O(\varepsilon)}} \text{ at new fixed point!}$$

$\Rightarrow O(\varepsilon)$ corrections to critical exponents

(A) Operator product expansions and RG flow equations

(A.1) The operator expansion



Bring two scaling operators/fields close enough:

$$\phi_i \phi_j \simeq \sum_n c_n S_n \quad \begin{matrix} \leftarrow \text{bare operators} \\ \text{around } \vec{r}_i \sim \vec{r}_j \end{matrix}$$

But each S_n can be decomposed onto the scaling fields ϕ_K !

We thus expect: $\phi_i(\vec{r}_1) \phi_j(\vec{r}_2) = \sum_K c_{ijk}(\vec{r}_1 - \vec{r}_2) \phi_K\left(\frac{\vec{r}_1 + \vec{r}_2}{2}\right)$

not important

- Valid when inserted in a correlation function $\langle \dots \Phi \rangle$ with $P \gg |\vec{r}_1 - \vec{r}_2|$
- Physically: ϕ_i, ϕ_j viewed from far away looks like a local quantity, which may be expanded on the ϕ_K 's.
- Strictly speaking, we should also include derivatives $\partial^\alpha \phi_K$ but not important for our purposes.

Using $\phi_i(\lambda r_i) = \lambda^{-\Delta_i} \phi(r_i)$:

$$c_{ijk}(\vec{r}_1 - \vec{r}_2) = \frac{c_{ijk}}{|\vec{r}_1 - \vec{r}_2|^{\Delta_i + \Delta_j - \Delta_k}}$$

↑
OPE coefficients

We will write

$$\phi_i \times \phi_j \sim \sum_K c_{ijk} \phi_K$$

- controls short distance (UV) singularities
- determines RG equations to second order

A.2 The perturbative RG

Consider:

$$S = S_0^* + \sum_i g_i \int \frac{d\vec{r}}{a^{d-\Delta_i}} \phi_i(\vec{r})$$

RG fixed point action/Hamiltonian

dimensionless couplings of S_0^*

$$[\phi_i] = L^{-\Delta_i}$$

Here we introduced $\int_{\vec{r}} = \int \frac{d\vec{r}}{a^d}$ with $a = \text{lattice spacing UV cutoff}$

Perturbative expansion assuming $g_i \ll 1$ and $Z_0 = \int D\phi e^{-S_0^*}$ "easy" / Known [3]

$$Z = Z_0 \left[1 - \sum_i g_i \int \frac{d\vec{n}_i}{a^{d-\Delta_i}} \langle \phi_i(\vec{n}) \rangle_0 + \frac{1}{2!} \sum_{i,j} g_i g_j \int \frac{d\vec{n}_i d\vec{n}_j}{a^{2d-\Delta_i-\Delta_j}} \langle \phi_i(\vec{n}_i) \phi_j(\vec{n}_j) \rangle_0 + \dots \right] \quad \text{and} \quad \langle \dots \rangle_0 = \frac{\int D[\phi] e^{-S_0^*} (\dots)}{Z_0}$$

$\mathcal{O}(g^3)$

We already adopted a continuum picture: lattice effects unimportant, But remember the lattice for short-distance (UV) divergences of integrals.

\Rightarrow regularize by requiring $|\vec{n}_i - \vec{n}_j| > a$ (Crude hard core cutoff to keep a continuum approach. End result won't depend on this)

Even with this UV cutoff a , perturbation theory near a critical point is still singular (cf previous chapters) \Rightarrow introduce an infrared cutoff L

All integrals are then finite : system in a finite box $V = L^d$

\hookrightarrow in this box: $\langle \phi_i(\vec{n}) \rangle_0 \neq 0$ and $\langle \phi_i(\vec{n}_i) \phi_j(\vec{n}_j) \rangle_0 \neq \frac{\delta_{ij}}{|\vec{n}_i - \vec{n}_j|^{2\Delta_i}}$

$$N = \frac{V}{a^d} \sim \# \text{ "sites"}$$

RG Scheme:

- Change the UV cutoff $a \rightarrow b a$ with $b = e^{\delta p} \approx 1 + \delta p$
- Keep L (or V) fixed so N decreases
- Ask how the couplings g_i should be changed to preserve Z (to order δp and $\mathcal{O}(g^2)$)

Two contributions: (will add up in β functions since $\delta P \ll 1$)

① $\frac{1}{a^{d-\Delta_i}}$ terms $\rightarrow \frac{1}{a^{d-\Delta_i}} (1 - (d - \Delta_i) \delta P)$ after rescaling

\Rightarrow can be compensated by changing $g_i \rightarrow g'_i = g_i + (d - \Delta_i) \delta P g_i$

First order =
dimensional
analysis

② Change in cutoff of integrals: $\int_{|\vec{n}_i - \vec{n}_j| > a(1+\delta P)} = \int_{|\vec{n}_i - \vec{n}_j| > a} - \int_{a(1+\delta P) > |\vec{n}_i - \vec{n}_j| > a}$
original contribution

Second term: use OPEs for $\delta P \ll 1$: $- \frac{1}{2} \sum_{ijk} C_{ijk} \int \frac{d\vec{n} d\vec{R}}{a^{2d-\Delta_i-\Delta_j}} \langle \phi_k(\vec{R}) \rangle_0 \frac{\eta^{\Delta_k}}{\eta^{\Delta_i+\Delta_j}}$
OPE coefficients
for S_d^*

integral over \vec{n} : $\int d\eta S_d \eta^{d-1-\Delta_i-\Delta_j+\Delta_k} \approx a^{d-\Delta_i-\Delta_j+\Delta_k} \delta P S_d$
 $a(1+\delta P) \nearrow 2\pi^{d/2}/\Gamma(d/2)$
area of unit hypersphere
in d dimension

$$\vec{\eta} = \vec{n}_i - \vec{n}_j$$

$$\vec{R} = \frac{\vec{n}_i + \vec{n}_j}{2}$$

from
OPE

This gives a contribution:

$$- \frac{1}{2} \sum_{ijk} C_{ijk} S_d \delta P \int \frac{d\vec{R}}{a^{d-\Delta_k}} \langle \phi_k(\vec{R}) \rangle_0$$

\Rightarrow can be compensated by changing $g_k \rightarrow g'_k = g_k - \frac{S_d}{2} \sum_{ij} C_{ijk} g_i g_j \delta P$

\Rightarrow Renormalized couplings:

$$g'_k = g_k + \delta P \left[\underbrace{(d - \Delta_k)}_{\substack{\text{RG eigenvalue at } S_d^* \\ \text{trivial fixed point } S_d^*}} g_k - \frac{S_d}{2} \sum_{ij} C_{ijk} g_i g_j \right]$$

non-trivial
one-loop "contribution"
to the β functions

in differential form:

$$\frac{dg_k}{dp} = \underbrace{(d - \Delta_k)}_{\gamma_k} g_k - \underbrace{\left(\frac{S_d}{2} \sum_{ij} C_{ijk} g_i g_j \right)}_{\text{not important: } g_i \rightarrow 2/\epsilon_i g_i} + \dots$$

OPE coefficients control the UV singularities of the integrals and hence the β functions. Higher order terms: 3 or more points within distance a in the integrals

(B) The ϕ^4 -theory (Ising model) in $d=4-\epsilon$ dimensions

Landau-Ginzburg action:

$$S = \int d^n \left[\frac{1}{2} (\nabla \phi)^2 + \frac{t}{a^2} \phi^2 + \frac{u}{a^{4-d}} \phi^4 \right]$$

ϕ normalized so that this coefficient is $= 1/2$

Leading order RG equations: $[\phi] = [L]^{\frac{2-d}{2}} \Rightarrow t$ and u dimensionless

$$\alpha \rightarrow \beta_\alpha : \quad t' = \beta^2 t \\ u' = \beta^{4-d} u$$

so that S invariant

$$\begin{aligned} \gamma_t &= 2 \\ \gamma_u &= 4-d \end{aligned}$$

at the fixed point
 $v = t = 0$

(B.1) Gaussian fixed point: $\beta = 1 + \delta\beta$:

$$\frac{dt}{d\beta} = 2t + \dots$$

$$\frac{du}{d\beta} = (4-d)u + \dots$$

$v = t = 0$ obvious fixed point:

$$S_0^* = \int d^n \frac{1}{2} (\nabla \phi)^2$$

Gaussian fixed point: invariant under RG

If $d > 4$: u is irrelevant, and there is one relevant (thermal) perturbation t (recall $t \sim |T-T_c|$ within Landau theory). We have $\gamma_t = 2$

$$\Rightarrow \gamma = 1/2 \quad \text{consistent with mean field} \quad (\gamma = 1/\gamma_t)$$

We also have $\Delta_\phi = \frac{d-2}{2}$ (by dimensional analysis) $\Rightarrow \langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle_0 \sim \frac{1}{|\vec{r}_1 - \vec{r}_2|^{d-2}} \Rightarrow \eta = 0$ on compute Gaussian integral

The fact that v is irrelevant for $d=4$ and the values of these exponents suggest that perhaps we could set $v=0$ and that the Ising model for $d \geq 4$ is controlled by a Gaussian fixed point + fermal perturbation $\propto \phi^2$

This is wrong: Add a field $-\frac{R}{a^{1+d/2}} \int d^d n \phi(n)$ to the action
(recall $[\phi] = L^{\frac{2-d}{2}}$)

$$\Rightarrow \gamma_R = \frac{d}{2} + 1 \text{ so that}$$

$$\begin{aligned}\beta &= \frac{d-2}{4} \\ \alpha &= 2 - d/2 \\ \gamma &= 0 \\ \delta &= \frac{d+2}{d-2}\end{aligned}$$

Exponents of Gaussian fixed point

vs Mean fields

\rightarrow satisfy hyperscaling

$$\begin{aligned}\beta &= 1/2 \\ \alpha &= 0 \\ \gamma &= 0 \\ \delta &= 3\end{aligned}$$

This agrees with MF only for $d=4$! This is because v is a dangerously irrelevant variable: we can't set it to 0!

- E.g.: ordered phase $\langle \phi \rangle \sim (-\frac{t}{v})^{1/2}$ by minimizing S : $v \neq 0$ needed!

- Gaussian theory controls fluctuations over MF results, but exponents are less singular for $d > 4$: MF contribution dominates: MF valid for $d > 4$

if $d < 4$: v is relevant

$$\frac{dv}{dp} = \epsilon v + \dots$$

with

$$\epsilon = 4 - d$$

\Rightarrow New fixed point (Wilson-Fisher): "close" to Gaussian fixed point if $\epsilon \ll 1 \Rightarrow$ controlled calculation in $d=4-\epsilon$ dimensions!
(in practice: set $\epsilon=1$ at the end of the calculation)

B.2) Wick theorem and OPEs of the Gaussian theory

All we need is the OPE structure of

$$S_0^* = \frac{1}{2} \int d^d \vec{r} (\nabla \phi)^2$$

Because $[\phi] = L^{\frac{2-d}{2}}$ $\Rightarrow \langle \phi(\vec{r}_1) \phi(\vec{r}_2) \rangle_0 = \frac{1}{|\vec{r}_1 - \vec{r}_2|^{d-2}}$ set to unity by redefining the normalization of ϕ

S_0^* : Higher order correlation functions given by Wick's theorem (See appendix on Gaussian integrals)

$$\langle \phi(r_1) \phi(r_2) \phi(r_3) \phi(r_4) \rangle_0 = \langle \phi(r_1) \phi(r_2) \rangle_0 \langle \phi(r_3) \phi(r_4) \rangle_0 + \langle \phi(r_1) \phi(r_3) \rangle_0 \langle \phi(r_2) \phi(r_4) \rangle_0 + \langle \phi(r_1) \phi(r_4) \rangle_0 \langle \phi(r_2) \phi(r_3) \rangle_0.$$

Also holds if some points coincide: $\langle \phi^2(r_1) \phi^2(r_2) \rangle_0 = 2 \langle \phi(r_1) \phi(r_2) \rangle_0^2 + \langle \phi^2 \rangle_0^2$ independent of $|\vec{r}_1 - \vec{r}_2|$

\Rightarrow it is convenient to use a slightly different basis:

$$\begin{aligned} :\phi^2: &= \phi^2 - \langle \phi^2 \rangle_0 \\ :\phi^3: &= \phi^3 - 3\langle \phi^2 \rangle_0 \phi \end{aligned}$$

"Normal ordered" scaling operators \Rightarrow no Wick contractions at the same point

\hookrightarrow work with these operators in S from the beginning

OPEs: $:\phi^2(r_1) : \phi^2(r_2) :$ = $\frac{2}{|\vec{r}_1 - \vec{r}_2|^{2d-4}} + \frac{4}{|\vec{r}_1 - \vec{r}_2|^{d-2}} :\phi^2(\vec{r}_1 + \vec{r}_2): + :\phi^4(\vec{r}_1 + \vec{r}_2): + \dots$

We write:

$$:\phi^2: \times :\phi^2: \sim 2 + 4 :\phi^2: + :\phi^4:$$

Other OPE's follow from simple combinatorics: (Focus on $\ell=0$: ϕ^{2n} operators only and ignore $n \geq 3$ for now) [8]

$$:\phi^2: x :\phi^4: \sim 4 \times 3 : \phi^2: + 8 : \phi^4:$$

$$:\phi^4: x :\phi^4: \sim 4 \times 3 \times 2 + \frac{(4 \times 3 \times 2)^2}{3!} : \phi^2: + \frac{(3 \times 4)^2}{2!} : \phi^4:$$

choose 3 on each side: $(4 \times 3 \times 2)^2$
choose $\binom{4 \times 3}{2}^2$ possibilities,
overcounts by 3!
(permutations of the indices)

We thus have:

$:\phi^2: x :\phi^2: \sim 2 + 4 : \phi^2: + : \phi^4:$
$:\phi^2: x :\phi^4: \sim 12 : \phi^2: + 8 : \phi^4:$
$:\phi^4: x :\phi^4: \sim 24 + 96 : \phi^2: + 72 : \phi^4:$

B.3 Wilson - Fisher Fixed point

We can immediately write down the RG flow equations:

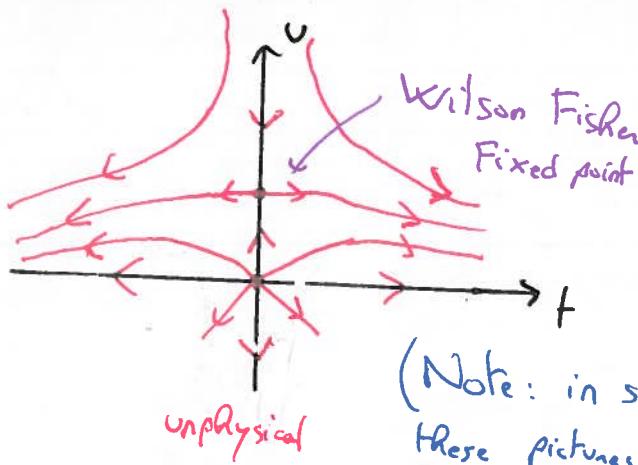
$$\frac{dt}{dp} = 2t - 4t^2 - 2 \times 12 t_0 - 96 u^2 + \dots$$

$$\frac{du}{dp} = \varepsilon u - t^2 - 2 \times 8 t_0 - 72 u^2 + \dots$$

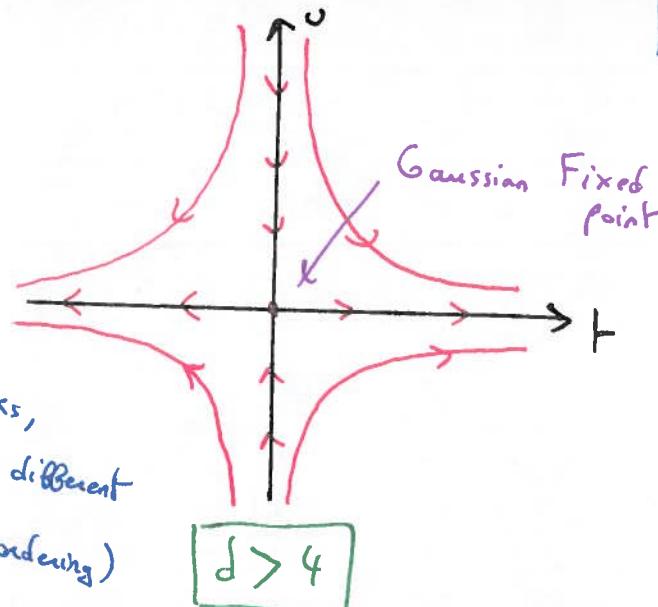
$C_{ijk} = C_{gik}$ if $i \neq j$

new fixed point
with: $t^* = 0 + \mathcal{O}(\varepsilon^2)$
 $u^* = \frac{\varepsilon}{72} + \mathcal{O}(\varepsilon^2)$

Thermal eigenvalue: $\frac{dt}{dp} = 2t - 24 u^* t + \dots \Rightarrow \gamma_t = 2 - \frac{24}{72} \varepsilon + \mathcal{O}(\varepsilon^2)$



$$d < 4$$



$$d > 4$$

(Note: in some books,
these pictures may look different
because of the normal ordering)

Using $\lambda = 1/\gamma_t$, we have

$$\lambda = \frac{1}{2} + \frac{\epsilon}{12} + \mathcal{O}(\epsilon^2)$$

Compute γ_t by introducing a field: $\phi \times \phi = 1 + : \phi^2 :$

$$\phi \times : \phi^2 : = 2\phi + : \phi^3 :$$

$$\phi \times : \phi^4 : = 4 : \phi^3 : + \dots$$

new operator: can be ignored to $\mathcal{O}(\epsilon)$. ϕ^3 is a redundant operator: can be absorbed by a shift $\phi \rightarrow \phi + C$
 \Rightarrow redefines t and u , with shift itself of order $\mathcal{O}(\epsilon)$
 (See e.g. Cardy)

$$\Rightarrow \frac{dR}{dp} = \left(\frac{d}{2} + 1 \right) R - 2 \times 2 R t + \dots$$

no uR term in $\frac{dR}{dp}$: $\eta = 0 + \mathcal{O}(\epsilon^2)$ ($\eta \neq 0$ for higher orders)

How useful is the ϵ expansion? Set $\epsilon = 1$ and $\epsilon = 2$ for $d = 3$ and $d = 2$

$$d=2: \eta_{\mathcal{O}(\epsilon)} = 0, \lambda_{\mathcal{O}(\epsilon)} = 0.66 \quad \left(\begin{array}{l} \text{exact: } \eta = 0.25, \lambda = 1 \\ \text{experiments: } \eta = 0.04, \lambda \approx 0.63 \end{array} \right)$$

$$d=3: \eta_{\mathcal{O}(\epsilon)} = 0, \lambda_{\mathcal{O}(\epsilon)} = 0.58 \quad \left(\begin{array}{l} \text{numerics} \end{array} \right)$$

\rightarrow not too bad for $d = 3$. Can be improved by going to $\mathcal{O}(\epsilon^5)$

$$d=2: \eta_{\mathcal{O}(\epsilon^5)} \approx 0.26, \lambda_{\mathcal{O}(\epsilon^5)} \approx 0.99$$

great agreement with numerics and experiments!

Requires extrapolations.
 ϵ -series asymptotic only

Remark: in retrospect, we only need two DPE coefficients:

$$:\phi^2: \times :\phi^4: = 12 :\phi^2: + \dots$$



Feynmann diagrams (not used here)

$$:\phi^4: \times :\phi^4: = 72 :\phi^4: + \dots$$



$\phi^2: \text{---} \quad (\text{not quite right})$
 $\phi^4: X \rightarrow \text{propagator}$

B.4

Irrelevant operators

: we can now justify why we could ignore terms like $\int d^d \eta g_{2n} \phi^{2n}$ in the action.

$$[\phi] = L^{-\frac{d-2}{2}}$$

$$[\phi^{2n}] = L^{n(2-d)}$$

$$\Delta_{\phi^{2n}} = n(d-2)$$

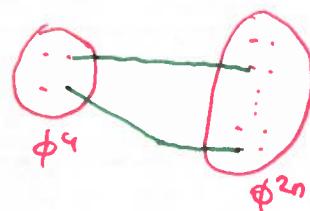
$$\Rightarrow \frac{dg_{2n}}{dp} = (d - n(d-2)) g_{2n} + \dots$$

near Gaussian Fixed point

if d is near 4, $\Delta_{\phi^{2n}} > 4$ for $n \geq 3$
 \Rightarrow irrelevance.

For $d \leq 4$, we should check that ϕ^6, ϕ^8, \dots are irrelevant at the WF fixed point:

$$:\phi^4: \times :\phi^{2n}: = \frac{4 \times 3 \times 2n(2n-1)}{2!} :\phi^{2n}: + \dots$$



$$\Rightarrow \frac{dg_{2n}}{dp} = (d - n(d-2)) g_{2n} - 2 \times 12n(2n-1) v g_{2n} + \dots$$

so that

$$\gamma_{2n} = d - n(d-2) - 24n(2n-1)v^* + \mathcal{O}(v^2)$$

$$= 4 - 2n + \varepsilon(n-1) - \underbrace{\frac{n(2n-1)}{3}\varepsilon}_{\text{correction due to } v^*} + \mathcal{O}(\varepsilon^2)$$

even more irrelevant
at Wilson Fisher fixed point

\Rightarrow contrary to real space approach, this Field theory ("continuum") calculation starting from Landau-Ginzburg is fully controlled!

(C) The $O(n)$ model in $d=4-\varepsilon$ dimensions

So far, we have focused on the universality class of the Ising Model.
 ⇒ To illustrate the generality of the approach, we now consider the $O(n)$ model

$$S = \int d^d \eta \left[\frac{1}{2} (\nabla \vec{\phi})^2 + \frac{1}{a^2} \vec{\phi}^2 + \frac{v}{a^{4-d}} (\vec{\phi}^2)^2 + \dots \right]$$

with $\vec{\phi}$: n component vector
 $(n=1: \text{Ising})$

C.1 ε expansion

Dimensional analysis is the same as Ising, and $\langle \phi_i \phi_j \rangle_0 = \frac{\delta_{ij}}{n^{d-2}}$ $i,j=1,\dots,n$
 at the Gaussian fixed point ($t=0, v=0$).

We need two OPE coefficients: $:\vec{\phi}^2: \times :(\vec{\phi}^2)^2: \sim \alpha : \vec{\phi}^2: + \dots$
 $:(\vec{\phi}^2)^2: \times :(\vec{\phi}^2)^2: \sim \beta (\vec{\phi}^2)^2 + \dots$

$\delta_{ij} \delta_{ik}$ index of $n: 2 \times 2 \times 2$

$$\sum_{i,j,k} : \phi_i \phi_i : \times : \phi_j \phi_j \phi_k \phi_k : \rightarrow \alpha = 4(n+2) \quad \begin{matrix} \text{using our result for} \\ n=1 \text{ (Ising)} \end{matrix}$$

$\delta_{ij} \cdot 4 \rightarrow 4n \text{ terms}$

$$\sum_{i,j,k,p} : \phi_i \phi_i \phi_j \phi_j : \phi_k \phi_k \phi_p \phi_p : \rightarrow \beta = 8(n+8) \quad \begin{matrix} \text{again using our results} \\ \text{for } n=1 \end{matrix}$$

$\delta_{ij} \text{ or } \delta_{ik}: 4n \text{ terms}$

$\delta_{jk} \text{ or } \delta_{jp}: 4n \text{ terms}$

$\delta_{ki} \text{ or } \delta_{ip}: 4n \text{ terms}$

⇒ This yields:

$$\boxed{\frac{du}{dp} = \varepsilon u - 8(n+8)u^2 + \dots} \rightarrow \text{fixed point at } u^* = \frac{\varepsilon}{8(n+8)}$$

$$\frac{dt}{dp} = 2t - 8(n+2)ut + \dots$$

$$\boxed{\gamma = \frac{1}{2} + \frac{1}{4} \frac{n+2}{n+8} \varepsilon + \mathcal{O}(\varepsilon^2)}$$

C.2 Large n limit of the $O(n)$ model

- For $n \rightarrow \infty$, $v^* \rightarrow 0$: suggests $\gamma_f = 2 - \varepsilon = d - 2$ exact in that limit

- Can be shown explicitly: change our notation slightly:

$$S = \frac{1}{2} \int d\vec{x} \left[(\nabla \vec{\phi})^2 + \Gamma \vec{\phi}^2 + \frac{U}{n} (\vec{\phi}^2)^2 \right]$$

$\frac{1}{n}$ factor here: needed as $n \rightarrow \infty$
(see below)

now use trick: $e^{-\frac{1}{2} \int d\vec{x} \frac{U}{n} (\vec{\phi}^2)^2}$

↑ interaction term

$$= \int D\sigma e^{-\frac{1}{2} \int d\vec{x} (\sigma^2 + i\sqrt{\frac{4U}{n}} \sigma \vec{\phi}^2)}$$

↑ $\vec{\phi}^2$: quadratic!

Hubbard-Stratonovich transformation
(up to unimportant constants)

Can be shown by completing the square:
(integral over σ Gaussian):

$$(\sigma + i\sqrt{\frac{U}{n}} \vec{\phi})^2 + \frac{U}{n} \vec{\phi}^2$$

$$\Rightarrow Z = \int D\sigma D\vec{\phi} e^{-\frac{1}{2} \int d\vec{x} \sigma^2} e^{-\frac{1}{2} \int d\vec{x} \sum_{i=1}^n \phi_i (-\nabla^2 + \Gamma + i\sqrt{\frac{4U}{n}} \sigma(x)) \phi_i}$$

we can integrate out ϕ : Gaussian integral!

$$= \int D\sigma e^{-\frac{1}{2} \int d\vec{x} \sigma^2} \left(\text{Det} \left[-\nabla^2 + \Gamma + i\sqrt{\frac{4U}{n}} \sigma \right] \right)^{-\frac{n}{2}}$$

n fields ϕ_i

Note: we're thinking (formally) of $-\nabla^2 + \Gamma + i\sqrt{\frac{4U}{n}} \sigma$ as an operator M that appears in the Gaussian integral: $\int d\vec{x} d\vec{x}' \phi_i(\vec{x}') M(\vec{x}', \vec{x}) \phi_i(\vec{x})$: "Matrix product"
with $M(\vec{x}', \vec{x}) = \delta(\vec{x}' - \vec{x}) (-\nabla^2 + \Gamma + i\sqrt{\frac{4U}{n}} \sigma(\vec{x}))$ with x = indices

eigenvalues

$$= \langle \vec{x}' | (-\nabla^2 + \Gamma + i\sqrt{\frac{4U}{n}} \sigma) | \vec{x} \rangle \quad \text{using QM notations}$$

$$\text{Det } M = \prod_K \lambda_K = \exp \left(\log \prod_K \lambda_K \right) = \exp \left(\sum_K \log \lambda_K \right) = \exp (T \eta \log M)$$

Rescaling $\sigma \rightarrow \sqrt{u} \sigma$, we get:

$$Z = \int D\sigma e^{-n S_{\text{eff}}[\sigma]}$$

$n \rightarrow \infty$

(path) integral dominated by saddle point

$$\text{with } S_{\text{eff}}[\sigma] = \frac{1}{2} \int d^d x \sigma^2(x) + \frac{1}{2} T n \log(-\nabla^2 + t + i\sqrt{4u}\sigma)$$

Effective action for auxiliary σ field

$$\text{Saddle point: } \delta S_{\text{eff}} = \int d^d x \sigma(x) \delta \sigma(x) + \frac{1}{2} T n \underbrace{\left[(-\nabla^2 + t + i\sqrt{4u}\sigma) \right]}_{M^{-1}} i\sqrt{4u} \delta \sigma$$

$$\int d^d x M^{-1}(x, x) i\sqrt{4u} \delta \sigma(x)$$

Here M^{-1} is the Green's function of $(-\nabla^2 + t + i\sqrt{4u}\sigma)$:

$$\frac{\delta S_{\text{eff}}}{\delta \sigma(x)} = 0 = \sigma + i\sqrt{u} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + t + i\sqrt{4u}\sigma} \quad \begin{array}{l} \text{where we assumed} \\ \sigma = \text{cst} \\ (\text{look for uniform} \\ \text{solution}) \end{array}$$

Let $t_{\text{eff}} = t + i\sqrt{4u}\sigma$ (assuming $\sigma \in i\mathbb{R}$)

$$\Rightarrow t_{\text{eff}} - t = 2u \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + t_{\text{eff}}} \quad \begin{array}{l} \text{UV cutoff: underlying lattice} \\ \text{equation for } t_{\text{eff}} \\ \text{A cut-off } k_a \end{array}$$

$$1/k^2 - \frac{t_{\text{eff}}}{K^2(K^2 + t_{\text{eff}})}$$

• repeating the mapping for the expectation value $\langle \vec{\phi}^2 \rangle$, one can show that there is a phase transition at $t_{\text{eff}} = 0$ (and $t_{\text{eff}} \sim \xi^{-2}$): if $t_{\text{eff}} < 0$, $\langle \vec{\phi}^2 \rangle$,

critical temperature $T = t_c = -2u \int \frac{d^d k}{(2\pi)^d} \frac{1}{K^2}$ \rightarrow convergent if $d > 2$
 critical T shifted down (vs Mean field: $t_c = 0$) \rightarrow (if $d < 2$: no transition:
 Mermin Wagner!)

$$\Rightarrow t_{\text{eff}} = t - t_c - 2u t_{\text{eff}} \int \frac{d^d k}{(2\pi)^d} \frac{1}{K^2(K^2 + t_{\text{eff}})}$$

$$\text{Now: } \int_0^\Lambda \frac{\frac{dK}{(2\pi)^d} \frac{1}{K^2(K^2 + t_{\text{eff}})}}{K^2(K^2 + t_{\text{eff}})} = t_{\text{eff}}^{d/2-2} \int_0^{\Lambda/\sqrt{t_{\text{eff}}}} \frac{dK \frac{S_d}{(2\pi)^d}}{K^2(K^2 + 1)}$$

$K \rightarrow \sqrt{t_{\text{eff}}} K$

$$= \begin{cases} t_{\text{eff}}^{d/2-2} \times \text{cst} & \text{if } d < 4 \\ t_{\text{eff}}^{d/2-2} \left(\frac{\Lambda}{\sqrt{t_{\text{eff}}}}\right)^{d-4} \sim \Lambda^{d-4} & \text{if } d > 4 \end{cases}$$

→ integral UV convergent
we can send $\Lambda \rightarrow \infty$

→ integral UV divergent
scales as Λ^{d-4} indep. of t_{eff}

- if $d > 4$: $t_{\text{eff}} \sim T - T_c$ and $t_{\text{eff}} \sim \xi^{-2}$ $\Rightarrow \beta = 1/2$ MF result ✓
- if $d < 4$: $t_{\text{eff}} \sim (T - T_c)^{\frac{2}{d-2}} \sim \xi^{-2}$ $\Rightarrow \beta = \frac{1}{d-2}$ consistent with ϵ expansion.
 $t_{\text{eff}} = T - T_c - \text{cst } t_{\text{eff}}^{\frac{d}{2}-1}$
negligible as $t_{\text{eff}} \rightarrow 0$ exact for $n \rightarrow \infty$
- Can be used as a starting point for the so-called large n expansion
 \Rightarrow compute corrections in $\frac{1}{n}$ and then set $n=1, 2, 3$ (usual physical values)

Appendix: Dangerously irrelevant variable in ϕ^4 theory for $d > 4$

. For $d > 4$, we expect fluctuations to be small and MF exponents to be correct
 $(\kappa = 0, \beta = 1/2, \gamma = 1, \delta = 3, \eta = 0, \zeta = 1/2)$

. The coupling $u\phi^4$ is irrelevant for $d > 4 \Rightarrow$ critical behavior controlled by Gaussian fixed point? (set $u = 0$) But some of the Gaussian theory exponents are different!

e.g. $\beta = \frac{d-2}{4}$ at Gaussian fixed point ($\gamma_t = 2, \gamma_R = 1 + \frac{d}{2}$)

\Rightarrow This is because $u\phi^4$ is **dangerously irrelevant**

RG equation for m : $m(t, R, u) = e^{n(\gamma_R - d)} F(t e^{n\gamma_t^2}, R e^{n\gamma_R}, u e^{n\gamma_u})$

set $R = 0, e^{n\gamma_t} t = O(1) \Rightarrow m(t, 0, u) = t^{-(\gamma_R - d)/\gamma_t} \Phi(u t^{-\gamma_u/\gamma_t})$

Naively: $t^{-\gamma_u/\gamma_t} \xrightarrow[t \rightarrow 0]{} 0$ if $d > 4$, and $\Phi(0)$ finite $\Rightarrow m \sim t^{-(\gamma_R - d)/\gamma_t}$
 (effectively: set $u = 0$) $\beta = \frac{d-2}{4}$

But: Landau theory predicts $m \propto \sqrt{-t}$ \Rightarrow can't set $u = 0$

$m(t, 0, u) \propto u^{-1/2} \Rightarrow \Phi(x) \underset{x \rightarrow 0}{\sim} x^{-1/2}$ limit $x \rightarrow 0$ ($u \rightarrow 0$)
 not smooth!

then $m(t, 0, u) = |t|^{-(\gamma_R - d)/\gamma_t} \Phi(|t|^{-\gamma_u/\gamma_t} u) \sim |t|^\beta$

with $\beta = \frac{d-\gamma_R}{\gamma_t} + \frac{\gamma_u}{2\gamma_t} = \frac{d-2}{4} + \frac{4-d}{4} = 1/2 \Rightarrow \boxed{\beta = 1/2}$

\Rightarrow we recover MF result as expected.