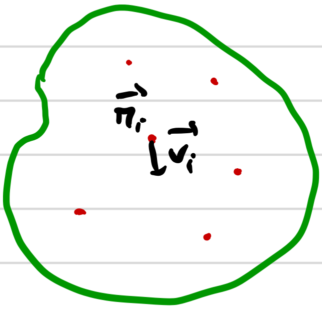
Newtonian Mechanics



System of particles S, isolated from the rest of the world.

Mechanics describes the evolution of S as a function of time: dynamics of the motion of the particles.

State $\hat{S}$ of the system: $\hat{S}(t) = \{ r_i(t), v_i(t) \}$

positions · velocities

where $\vec{v}_i = \frac{d\vec{r}_i}{dt} \equiv \dot{\vec{r}}_i$

I Newton's laws

1st law: A body remains at rest or in uniform motion unless acted upon by a force.

$\Rightarrow$ if $\sum_j \vec{F}_{ij} = \vec{0}$ with $\vec{F}_{ij}$ force exerted by particle j on particle i, then the i^{th} particle has acceleration $= 0$: uniform motion or rest

2nd law: $\boxed{m_i \ddot{\vec{r}}_i = \sum_j \vec{F}_{ij}}$

$\vec{F}_{ij} = \vec{F}_{j \to i}$

Equations of motion

m_i: positive number: inertial mass

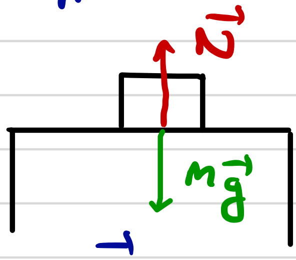
$\vec{a}_i = \frac{d\vec{v}_i}{dt} = \ddot{\vec{r}}_i \Rightarrow m\,\vec{a}_i = \vec{F}_i = \sum_{j=1}^{N} \vec{F}_{ij}$

N particles

3rd law: If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

$$\boxed{\vec{F}_{ij} = -\vec{F}_{ji}}$$

ex:



$\vec{N} = -m\vec{g}$

Consequence: $\sum_i m_i \ddot{\vec{r}}_i = \sum_{i,j} \vec{F}_{ij} = \vec{0}$

$\Rightarrow \dfrac{d}{dt}\left(\sum_i m_i \vec{v}_i\right) = \vec{0}$: total linear momentum conserved

$\boxed{\vec{P}_i = m_i \vec{v}_i}$ momentum of particle i

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* These laws are approximate, they break down when:

  - For velocities near the speed of light (⇝ special relativity)
  - For atomic and subatomic systems (⇝ quantum mechanics)
  - When gravity is strong (⇝ general relativity)

* In the following, we will assume that:

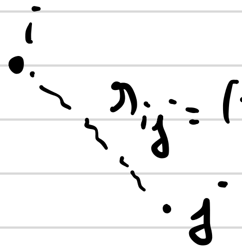
  - Newton's equations hold!
  - $\vec{F}_{ij} = F(\vec{r}_i, \vec{r}_j)$: function of $\vec{r}_i, \vec{r}_j$ only
  - Conservative forces :
    Potential

$\vec{\nabla}_{\vec{r}_i}$: gradient

$$\boxed{\vec{F}_{ij} = -\frac{\partial}{\partial \vec{r}_i} U(\vec{r}_i, \vec{r}_j)}$$

in principle, can also depend on i, j

- Finally, we will require that the interaction potential depends only on distance:



$r_{ij} := |\vec{r}_i - \vec{r}_j|$ $\qquad U(\vec{r}_i, \vec{r}_j) = U(\vec{r}_j, \vec{r}_i) \equiv U(r_{ij})$

---> central forces

This implies: $\vec{F}_{ij} = -\frac{\partial}{\partial \vec{r}_i} U(|\vec{r}_i - \vec{r}_j|) = -\frac{\partial |\vec{r}_i - \vec{r}_j|}{\partial \vec{r}_i} U'(r_{ij})$

$$= -\frac{\partial \sqrt{(\vec{r}_i - \vec{r}_j)^2}}{\partial \vec{r}_i} U'(r_{ij})$$

$$= -\frac{1}{2|\vec{r}_i - \vec{r}_j|} 2(\vec{r}_i - \vec{r}_j) U'(r_{ij})$$

Let: $\varphi(r) = -\frac{1}{r}\frac{dU}{dr} \implies \boxed{\vec{F}_{ij} = (\vec{r}_i - \vec{r}_j)\varphi(r_{ij})}$

Note: $\vec{F}_{ij} = -\vec{F}_{ji}$ (3^{rd} law!) and $\vec{F}_{ij}$ parallel to $\vec{r}_i - \vec{r}_j$

Inertial frame: Motion measured with respect to reference frame.

Inertial frame: frame in which Newton's laws are valid.
"fixed" stars frame

Galilean invariance : if Newton's law are valid in one reference frame, then they are also valid in any reference frame in uniform motion (not accelerated) with respect to the first one.

Follows from : $m_i \ddot{\vec{x}}_i = \vec{F}_i$

$\vec{x}\,' = \vec{x} + \vec{v}_0 t \qquad m_i \ddot{\vec{x}}_i\,' = \vec{F}_i$

Galilean Boost (3)
+ Rotations (3)
+ translations (3)
+ $t' = t + c$ (1)
= Galilean Group

II Conservation theorems

Momentum : Reminder: 3^{rd} law: $\vec{F}_{12} = -\vec{F}_{21}$

$$m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt} \Rightarrow \frac{d}{dt}\Big(\underbrace{m_1\vec{v}_1 + m_2\vec{v}_2}_{\text{Momentum : conserved!}}\Big) = 0$$

. If $\vec{F} = 0$, $\vec{p}$ is a constant of motion.

. If $\vec{F}.\vec{a} = 0$ for a given constant vector $\vec{a}$

then $\frac{d\vec{p}}{dt}.\vec{a} = \frac{d(\vec{p}.\vec{a})}{dt} = 0 \Rightarrow \vec{p}.\vec{a}$ conserved

Ex : Particle in magnetic field $\vec{B}$: $\vec{F} = q\,\vec{v} \times \vec{B}$

$\vec{F}.\vec{B} = 0$ since $\vec{F} \perp \vec{B}$

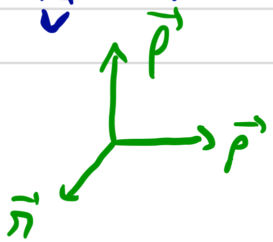
$\Rightarrow \vec{p}.\vec{B}$ conserved! $m v_z$ along $\vec{B} = B\vec{e}_z$ constant.

Angular Momentum : $\boxed{\vec{L} = \vec{r} \times \vec{p}}$ with respect to origin from which $\vec{r}$ is measured. $\vec{v} = \dot{\vec{r}}$

$p = m\vec{v}$

. $\|\vec{L}\| = r v \sin\theta \qquad \vec{L} = 0$ if $\vec{r} \,/\!/\, \vec{v}$

. By definition $\vec{L}$ is $\perp$ to $\vec{r}$ and $\vec{p}$



. Let's also introduce the TORQUE (moment of force)

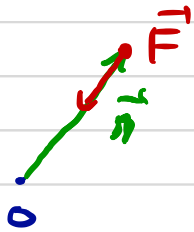
$$\boxed{\vec{T} = \vec{r} \times \vec{F}}$$

Now: $$\frac{d\vec{p}}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{\vec{v} \times \vec{p} = m\, \vec{v} \times \vec{v} = 0} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$ 2nd law
$$= \vec{r} \times \vec{F}$$
$$= \vec{T}$$

$$\Rightarrow \quad \boxed{\frac{d\vec{p}}{dt} = \vec{T}}$$

The angular momentum of a particle subject to no torque is conserved.

Ex: Central Force:



$\vec{F} = -F\, \vec{e_r} \Rightarrow \vec{T} = \vec{0}$

$\vec{e_r} = \frac{\vec{r}}{\|\vec{r}\|}$

Energy: Def: Work: $$\boxed{W_{1\to 2} = \int_1^2 d\vec{r} \cdot \vec{F}}$$

Line integral following path from 1 to 2.

$$m \frac{d\vec{v}}{dt} = \vec{F} \quad \Rightarrow \quad m \frac{d\vec{v}}{dt} \cdot d\vec{r} = \vec{F} \cdot d\vec{r}$$
← total force

$$\Rightarrow \quad W_{1\to 2} = \int_1^2 \vec{F} \cdot d\vec{r} = \int_1^2 m \frac{d\vec{v}}{dt} \frac{d\vec{r}}{dt}\, dt$$

$$= \int_{t_1}^{t_2} m\, \vec{v}\cdot\frac{d\vec{v}}{dt}\, dt = \int_{t_1}^{t_2} m \frac{d}{dt}\left(\frac{\vec{v}\cdot\vec{v}}{2}\right) dt$$

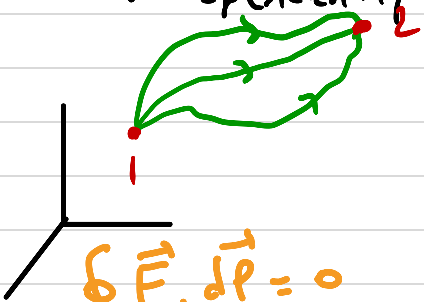
$$= \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt = \int_1^2 d\left(\frac{1}{2} m v^2\right)$$

$$= T_2 - T_1 \quad \text{where} \quad \boxed{T = \frac{1}{2} m v^2}$$ Kinetic energy

So we have $\boxed{\Delta T = T_2 - T_1 = W_{1\to 2}}$ Kinetic energy theorem.

Variation of kinetic energy between 1 and 2 given by the work of the forces exerted on the particle.

Def: Conservative Force: Force for which $W_{1\to 2} = U_1 - U_2$ independently of the path chosen : $\vec{\nabla}\times\vec{F} = 0$



$\oint \vec{F}\cdot d\vec{l} = 0$

$$\vec{F} = -\vec{\nabla} U$$ also V : potential energy, $\nabla = \frac{\partial}{\partial \vec{r}}$

$$W_{1\to 2} = -\int_1^2 \vec{\nabla} U \cdot d\vec{r} = -\int_1^2 dU = U_1 - U_2$$

Note: U is defined up to an additive constant.

Assuming a conservation force:

$$dT = d\left(\frac{1}{2} m v^2\right) = \vec{F}\cdot d\vec{r} \Rightarrow \frac{dT}{dt} = \vec{F}\cdot\vec{v} = -\vec{\nabla} U\cdot\dot{\vec{r}}$$

and $$\frac{dU}{dt} = \sum_i \frac{\partial U}{\partial x_i}\frac{dx_i}{dt} + \frac{\partial U}{\partial t} = \vec{\nabla} U\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t}$$

$$\Rightarrow \frac{d}{dt}(T+U) = \frac{\partial U}{\partial t}$$

For a conservative force, we also require $U(\cancel{t})$

$$\Rightarrow \boxed{\frac{d}{dt}(T+U) = 0}$$

$\boxed{E = T+U}$ Energy conserved for a conservative force

III Conservative forces in 1d

In 1d, any force that depends on x only (not on time) is conservative:

$$F(x) \Rightarrow \text{choose } U(x) = -\int_{x_0}^{x} F(x')\,dx'$$

so $F = -U'(x)$

$$m\frac{dv}{dt} = -U'(x) \Rightarrow m v \frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\frac{dU}{dx}\frac{dx}{dt}$$
$$= -\frac{dU}{dt}$$

$$\boxed{\frac{1}{2}mv^2 + U = E}$$

(total) energy conserved.

E = cst fixed by initial conditions

. Separate variables: $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = E - U(x)$

$$\Rightarrow \pm\int_{x_0}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}(E - U(x'))}} = \int_{t_0}^{t} dt' = t - t_0$$

Ex: Harmonic oscillator, let's use energy conservation to solve that problem.

$$F = -kx = -\frac{d}{dx}\underbrace{\left(\frac{1}{2}kx^2\right)}_{U}$$ $x=0$ = equilibrium position

$t_0 = 0$

$$\Rightarrow t = \int_{x_0}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}\left(E - \frac{1}{2}kx'^2\right)}}$$ with $E = \frac{1}{2}kx_0^2$ (choose $v_0 = 0$)

$$\Rightarrow t = \int_{x_0}^{x(t)} \frac{1}{\sqrt{k/m}} \frac{dx'}{\sqrt{x_0^2 - x'^2}} = \sqrt{\frac{m}{k}} \int_1^{x(t)/x_0} \frac{dy}{\sqrt{1-y^2}}$$ $y = x/x_0$

$$= \sqrt{\frac{m}{k}} \Big[-\text{Arccos}(y)\Big]_1^{x/x_0}$$ $\left(\text{Arccos}'(x) = -1/\sqrt{1-x^2}\right)$

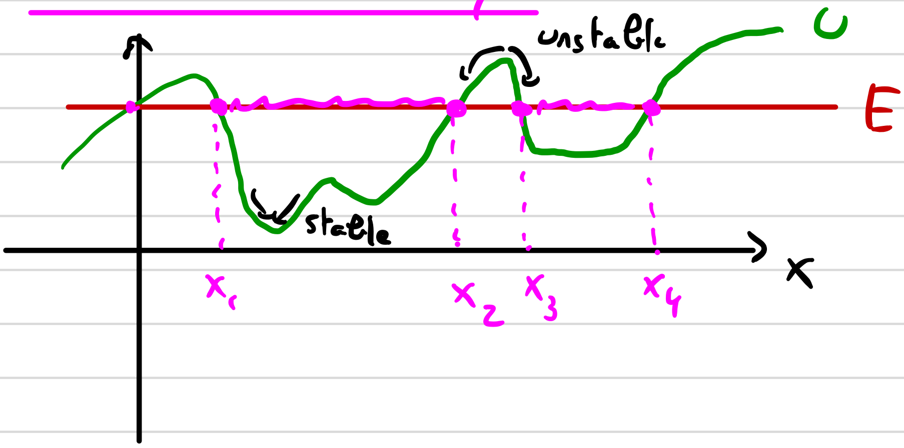
$$= -\sqrt{\frac{m}{k}}\ \text{arccos}\left(x/x_0\right)$$

Note: $\sin^2(\arccos x) + x^2 = 1 \Rightarrow 2(\arccos' x)\sin(\arccos x)x + 2x = 0$

$\Rightarrow \arccos' x = -1/\sin(\arccos x) = -1/\sqrt{1-x^2}$

$$\Rightarrow x(t) = x_0 \cos\omega t \quad \text{with } \omega = \sqrt{\frac{k}{m}}.$$ ✓

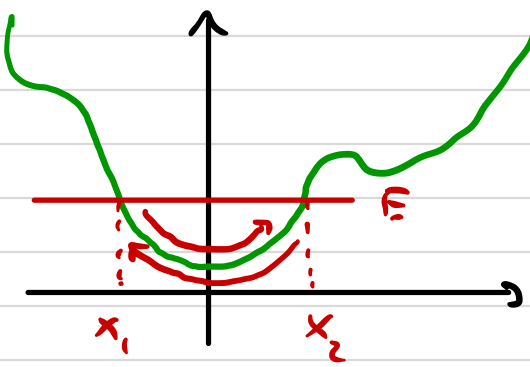
* Qualitative analysis:



- Since $E = T + U$ with $T > 0$, motion can only happen for $E > U$: between x_1 and x_2, x_3 and x_4 ...

At $x_1, x_2 \ldots$: $T = 0$ since $U = E$: turning points
v vanishes.

• Motion between x_1 and x_2 is periodic.



Period: 2 trips between x_1 and x_2

$$T = 2\int_{x_1}^{x_2} \frac{dx'}{\sqrt{\frac{2}{m}(E-U(x'))}}$$

Ex: Harmonic Oscillators: $U(x) = \frac{1}{2}kx^2$ $\quad$ x_1 and x_2 given by

$$E = \frac{1}{2}kx_{1,2}^2 \Rightarrow x_{1,2} = \pm\sqrt{\frac{2E}{k}}$$

$$T = 2\int_{-\sqrt{2E/k}}^{\sqrt{2E/k}} \frac{dx'}{\sqrt{\frac{k}{m}}\sqrt{x_{1,2}^2 - x'^2}} = 2\sqrt{\frac{m}{k}}\int_{-1}^{1}\frac{dy}{\sqrt{1-y^2}} = \frac{2\pi}{\omega} \checkmark$$

$y = x/x_2$

$$\underbrace{-\arccos 1}_{0} + \arccos(-1) = \pi$$

IV Many particles

2nd law: $\dot{\vec{p}}_i = \vec{F}_i = \vec{F}_i^{ext} + \sum_{j\neq i}\vec{F}_{j\to i}$

$$\sum_i \dot{\vec{p}}_i = M\ddot{\vec{R}} = \sum_i \vec{F}_i^{ext} + \underbrace{\sum_i\sum_{j\neq i}\vec{F}_{j\to i}}_{\sum_{i<j}(\vec{F}_{i\to j} + \vec{F}_{j\to i}) = \vec{0}}$$

with $M = \sum_i m_i$, $\boxed{\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i}$ Center of Mass

$$\vec{F}^{ext} = \vec{0} \quad \Rightarrow \quad \vec{P} \text{ conserved}$$

Angular momentum revisited

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \underbrace{\sum_{i \neq j} \vec{r}_i \times \vec{F}_{j\to i}}_{\sum_{i<j} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{j \to i}} + \underbrace{\sum_i \vec{r}_i \times \vec{F}_i^{ext}}_{\text{external torque}}$$

$= 0$ if $\vec{F}_{j\to i}$ parallel to $\vec{r}_i - \vec{r}_j$

Let $\vec{r}_i = \vec{R} + \vec{\delta}_i$: $\quad \vec{L} = \sum_i m_i (\vec{R} + \vec{\delta}_i) \times (\dot{\vec{R}} + \dot{\vec{\delta}}_i)$

$$= M \vec{R} \times \dot{\vec{R}} \rightarrow \text{orbital}$$

$$\left.\begin{aligned} &+ \sum_i m_i \vec{R} \times \dot{\vec{\delta}}_i \\ &+ \sum_i m_i \vec{\delta}_i \times \dot{\vec{R}} \end{aligned}\right\} \vec{0}: \quad \sum_i m_i \vec{\delta}_i = \sum_i m_i (\vec{r}_i - \vec{R}) = \sum_i m_i \vec{r}_i - M\vec{R} = \vec{0}$$

$$+ \sum_i m_i \vec{\delta}_i \times \dot{\vec{\delta}}_i \rightarrow \vec{\ell}_{int} : \text{"spin"}$$

Energy revisited

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}$$

$$\sum_i m_i \dot{\vec{\delta}}_i = 0$$

$$\vec{r}_i = \vec{R} + \vec{\delta}_i : \quad T = \sum_i \frac{1}{2} m_i \left(\dot{\vec{R}}^2 + \dot{\vec{\delta}}_i^{\,2} + \cancel{2 \dot{\vec{R}} \cdot \dot{\vec{\delta}}_i} \right)$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum_i m_i \dot{\vec{\delta}}_i^{\,2}$$

$$E = T + U \quad \text{conserved with} \quad U = \sum_i U_i^{ext} + \sum_{i<j} U_{ij}$$

Proof: $$dT = \sum_i d\left(\frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}\right) = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d\dot{\vec{r}}_i}{dt}\, dt$$

$$= \sum_i \left(\vec{F}_i^{ext} + \sum_{j\neq i} \vec{F}_{j\to i}\right) \cdot d\vec{r}_i$$

$$= \sum_i -\vec{\nabla} U_i^{ext} \cdot d\vec{r}_i \; - \sum_{i<j} \underbrace{\vec{F}_{j\to i}}_{-\frac{\partial}{\partial(\vec{r}_i - \vec{r}_j)} U_{ij}(\|\vec{r}_i - \vec{r}_j\|)} \cdot (d\vec{r}_i - d\vec{r}_j)$$

$$= -dU$$

We'll come back to rigid bodies later.

$\|\vec{r}_i - \vec{r}_j\|$ fixed

(V) Rest of the course

Euler, Lagrange, Hamilton and Jacobi: more formal formulation of mechanics.

and more elegant!

- → Easier to tackle more complex problems
- → More transparent: symmetries, mathematical structure
- → Lagrangian / Hamiltonian formalism underlies $\underline{\underline{\text{all}}}$ (!) of modern physics (QFT, stat mech, GR etc...)
- → Connections to QM, Stat Mech...