Variational Calculus

• 2^{nd} law: $m\ddot{\vec{r}} = -\vec{\nabla} V$ for a single body of mass m, in a conservative force $\vec{F} = -\vec{\nabla} V$

or in components $m\dfrac{d^2 r_i}{dt^2} = -\dfrac{\partial V}{\partial r_i}$ (here: $i = x, y, z$)

now write: $m\dot{r}_i = m v_i = \dfrac{\partial}{\partial v_i}\left[\frac{1}{2} m \vec{v}^2\right] = \dfrac{\partial}{\partial v_i}\left[\frac{1}{2} m \sum_{j=x,y,z} v_j^2\right]$

$$\Rightarrow \quad \frac{d}{dt}\left[\frac{\partial}{\partial \dot{r}_i}\left(\frac{1}{2} m \vec{v}^2\right)\right] = -\frac{\partial V}{\partial r_i}$$

Lagrangian

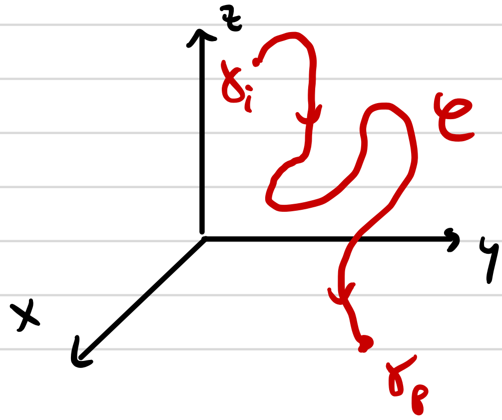
Let $\boxed{L = T - V} = \frac{1}{2} m \vec{v}^2 - V(r)$

then: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}_i}\right) - \dfrac{\partial L}{\partial r_i} = 0$

$T = T(\dot{r}_i)$
$V = V(r_i)$

We'll now see how this equations emerges in much more general contexts.

I Functionnals and variational calculus



Curve: $(x(\gamma), y(\gamma), z(\gamma))$

parametrized by $\gamma \in [\gamma_i, \gamma_f]$

Functional: Function $f(x) \longmapsto F[f] \in \mathbb{R}$ "Number"

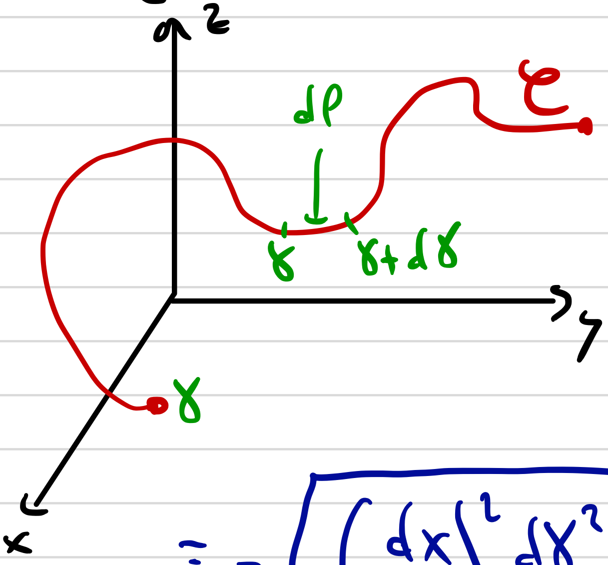
Ex: $F[f] = \int_0^1 f(x)\,dx$ for every function defined on $[0,1]$

$$F[f(x)=x] = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$F\left[f(x) = x^3/2\right] = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{8}$$

Ex: Curve $\mathcal{C}(\gamma) = (x_\gamma, y_\gamma, z_\gamma)$ w/ $x_\gamma = x(\gamma)$ etc.

Length of $\mathcal{C}$ = functional of $\mathcal{C}(\gamma)$



$$d\ell = \|\vec{r}_{\gamma+d\gamma} - \vec{r}_\gamma\|$$

$$= \sqrt{(x_{\gamma+d\gamma} - x_\gamma)^2 + (y_{\gamma+d\gamma} - y_\gamma)^2 + (z_{\gamma+d\gamma} - z_\gamma)^2}$$

$$= \sqrt{\left(\frac{dx}{d\gamma}\right)^2 d\gamma^2 + \left(\frac{dy}{d\gamma}\right)^2 d\gamma^2 + \left(\frac{dz}{d\gamma}\right)^2 d\gamma^2}$$

$$= d\gamma \sqrt{\sum_{i=1}^{d}\left(\frac{dx_i}{d\gamma}\right)^2}$$

$x_i = x, y, z$ for $d = 3$, $i = 1, 2, 3$

Length:

$$\boxed{\mathcal{L} = \int_{\mathcal{C}} d\ell = \int_{\gamma_i}^{\gamma_f} d\gamma \sqrt{\sum_{i=1}^{d}\left(\frac{dx_i}{d\gamma}\right)^2}}$$

Variational Calculus:

Goal: find function(s) that maximize(s) (minimize(s)) a certain functional.

eg: Which curve gives the shortest path between two points?

$$\underbrace{I}_{\text{Functional}} = \int_{\gamma_i}^{\gamma_f} \underbrace{F}_{\text{Function!}}\left(\vec{x}(\gamma), \vec{x}'(\gamma)\right) d\gamma \qquad \vec{x} = (x_1, \dots, x_d), \quad i = 1, \dots, d.$$

- Just like to extremize a function $f(x)$, we find x such that $f'(x) = 0$ (derivative $= 0$), here we will extremize $I[\vec{x}(\gamma)]$ such that

$$\underbrace{\frac{\delta I}{\delta x_i(\gamma)}}_{= \text{ Functional derivative}} = 0 \qquad i = 1, \dots, d$$

$$\tilde{\vec{x}}(\gamma) = \vec{x}(\gamma) + \delta\vec{x}(\gamma) \qquad = \sum_i \frac{\delta I}{\delta x_i} \delta x_i(\gamma)$$

$$\boxed{I\left[\tilde{\vec{x}}(\gamma)\right] = I\left[\vec{x}(\gamma)\right] + \int d\gamma \, \underbrace{\frac{\delta I}{\delta \vec{x}(\gamma)}}_{\text{definition of } \frac{\delta I}{\delta x(\gamma)}} \cdot \delta\vec{x}(\gamma) + \mathcal{O}(\delta\vec{x}^2)}$$

"Taylor expansion" for functionals!

We also have:

$$\boxed{\frac{\delta I}{\delta x_i(\gamma)} = \lim_{\delta x_i(\gamma) \to 0} \frac{I\left[\vec{x}(\gamma) + \delta\vec{x}(\gamma)\right] - I\left[\vec{x}(\gamma)\right]}{\delta x_i(\gamma)}}$$

Let's compute this explicitly:

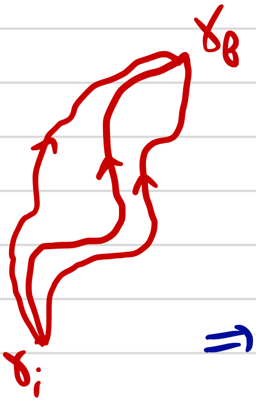
$$I[\vec{\tilde{x}}(\gamma)] = \int_{\gamma_i}^{\gamma_f} F\left[\vec{x}(\gamma) + \delta\vec{x}(\gamma),\ \vec{x}'(\gamma) + \delta\vec{x}'(\gamma)\right] d\gamma$$

Taylor expand

$$\overset{\downarrow}{=} \int_{\gamma_i}^{\gamma_f} \Big(F[\vec{x}(\gamma), \vec{x}'(\gamma)] + \underbrace{\frac{\partial F}{\partial \vec{x}(\gamma)} \cdot \delta\vec{x}(\gamma)}_{\sum_{i=1}^{d} \frac{\partial F}{\partial x_i(\gamma)} \delta x_i(\gamma)} + \frac{\partial F}{\partial \vec{x}'(\gamma)} \cdot \delta\vec{x}'(\gamma) + \mathcal{O}(\delta\vec{x}^2) \Big) d\gamma$$

$$= I[\vec{x}(\gamma)] + \int_{\gamma_i}^{\gamma_f} d\gamma \left[\frac{\partial F}{\partial \vec{x}(\gamma)} \cdot \delta\vec{x}(\gamma) + \frac{\partial F}{\partial \vec{x}'(\gamma)} \cdot \frac{d}{d\gamma} \delta\vec{x}(\gamma) \right] + \ldots$$

integrate by parts



Boundary term $= 0$ if we assume $\delta\vec{x}(\gamma_i) = \delta\vec{x}(\gamma_f) = 0$

$$\Rightarrow I[\vec{\tilde{x}}] - I[\vec{x}] = \int_{\gamma_i}^{\gamma_f} d\gamma \left[\frac{\partial F}{\partial \vec{x}} - \frac{d}{d\gamma}\left(\frac{\partial F}{\partial \vec{x}'} \right) \right] \cdot \delta\vec{x}(\gamma) + \ldots$$

$$\Rightarrow \frac{\delta I}{\delta \vec{x}(\gamma)} = \frac{\partial F}{\partial \vec{x}} - \frac{d}{d\gamma}\left(\frac{\partial F}{\partial \vec{x}'} \right) \qquad \left(\vec{x}' = \frac{d}{d\gamma}\vec{x} \text{ here} \right)$$

$I[\vec{x}(\gamma)]$ is extremized by $\vec{x}(\gamma)$ satisfying:

$$\boxed{\frac{\partial F}{\partial x_i}\left(\vec{x}(\gamma), \vec{x}'(\gamma)\right) = \frac{d}{d\gamma} \frac{\partial F}{\partial x_i'}\left(\vec{x}(\gamma), \vec{x}'(\gamma)\right)} \qquad i = 1, \ldots, d$$

Euler-Lagrange Equations

Two-line version (single variable):

$$\delta\left(\int d\gamma\, F(x,x')\right) = \int d\gamma \left(\frac{\partial F}{\partial x}\,\delta x + \frac{\partial F}{\partial x'}\,\underbrace{\delta x'}_{\frac{d}{d\gamma}\delta x}\right)$$

$$= \int d\gamma\, \underbrace{\left(\frac{\partial F}{\partial x} - \frac{d}{d\gamma}\left(\frac{\partial F}{\partial x'}\right)\right)}_{\frac{\delta I}{\delta x(\gamma)}}\, \delta x(\gamma)$$

(II) An important theorem:

Let $E = \vec{x}'\cdot\frac{\partial F}{\partial \vec{x}'} - F$. Then $\boxed{\frac{dE}{d\gamma} = -\frac{\partial F}{\partial \gamma}}$

with $\vec{x}(\gamma)$ such that $\delta I = 0$

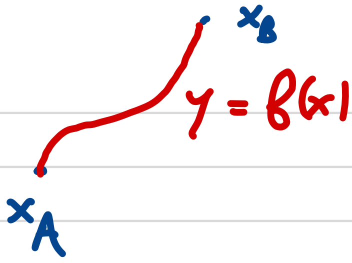
if $\frac{\partial F}{\partial \gamma} = 0$: $E = \text{Const}$ (first integral of the Euler-Lagrange equations)

Proof: $\frac{dE}{d\gamma} = \frac{d}{d\gamma}\left(\vec{x}'\cdot\frac{\partial F}{\partial \vec{x}'}\right) - \frac{dF}{d\gamma}$

$$= \vec{x}''\cdot\frac{\partial F}{\partial \vec{x}'} + \vec{x}'\cdot\underbrace{\frac{d}{d\gamma}\left(\frac{\partial F}{\partial \vec{x}'}\right)}_{\frac{\partial F}{\partial \vec{x}}} - \left(\frac{\partial F}{\partial \gamma} + \frac{\partial F}{\partial \vec{x}}\cdot\vec{x}' + \frac{\partial F}{\partial \vec{x}'}\cdot\vec{x}''\right)$$

$$= -\frac{\partial F}{\partial \gamma}$$

Example



Length of curve:

$$I = \int \sqrt{dx^2 + dy^2} = \int dx \sqrt{1 + (f')^2}$$

$$F(f, f') = \sqrt{1 + (f')^2} \quad , \text{ indep of } f \text{ and } x$$

$$E = f' \frac{\partial F}{\partial f'} - F = \text{const} \implies f' = \text{const}$$

$$\implies f = Ax + B$$

A, B fixed by boundary conditions.

Shortest path = straight line

Euler Lagrange: $\frac{d}{dx}\left(\frac{\partial F}{\partial f'}\right) = \frac{\partial F}{\partial f} = 0 \implies \frac{\partial F}{\partial f'} = \text{const.}$

III Constraints

(intro: we'll come back to this topic in the next chapter)

Example: (without functionals). Suppose we want to extremize (maximize in this case) $f(x,y) = x + y$ subject to the constraint $x^2 + y^2 = 1$.

Introduce:

$$F(x, y, \lambda) = x + y + \lambda (x^2 + y^2 - 1)$$

Lagrange multiplier

Extremize F: $\partial_x F = 1 + 2x\lambda = 0$

$$\partial_y F = 1 + 2y\lambda = 0$$

$$\partial_\lambda F = x^2 + y^2 - 1 = 0 \quad : \text{ constraint!}$$

$$\Rightarrow \quad x = y = -\frac{1}{2\lambda} \text{ and } \frac{1}{2\lambda^2} = 1 \Rightarrow \lambda = \pm 1/\sqrt{2}$$

$$\Rightarrow \quad x = y = \pm \frac{\sqrt{2}}{2}, \quad \begin{array}{l} + \text{ sign : maximum} \\ - \text{ sign : minimum.} \end{array}$$

<u>More generally</u> : Functional $I[\vec{x}] = \int d\gamma\, F(\vec{x}, \vec{x}')$

\+ constraints : ① $Q_i[x] = \int d\gamma\, q_i(\vec{x}(\gamma)) = 0$

$i = 1, \dots, \#$ constraints

② $q_i(\vec{x}(\gamma)) = 0, \quad \forall \gamma.$

Modified functional : ① $\tilde{I} = I + \sum_i \lambda_i Q_i = I[x, \vec{\lambda}]$

② $\tilde{I} = \int d\gamma \left(F + \sum_i \lambda_i(\gamma)\, q_i(\vec{x}(\gamma)) \right)$

function of γ in case 2

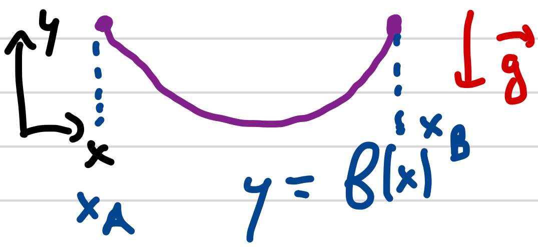
Euler Lagrange equations: $\tilde{F} = F + \sum_i \lambda_i q_i$

$$\frac{\partial}{\partial x}\left(F + \sum_i \lambda_i q_i\right) = \frac{d}{d\gamma}\left(\frac{\partial F}{\partial x'}\right) \Rightarrow \frac{d}{d\gamma}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = \sum_i \lambda_i q_i'$$

① $\dfrac{dI}{d\lambda_i} = 0 \Rightarrow Q_i = 0$: λ_i not a function here

② $\dfrac{\partial}{\partial \lambda_i'} \tilde{F} = 0 \Rightarrow \dfrac{\partial \tilde{F}}{\partial \lambda_i(\gamma)} = q_i(\gamma) = 0$ (constraints)

Example



flexible cable suspended by two points.

(Catenary problem)

Minimize potential energy : $E = + g \underset{\text{mass/length}}{\mu} \int y \, d\ell$

forget units : minimize $I = \displaystyle\int_{x_A}^{x_B} dx \, f \sqrt{1+(f')^2}$

with constraint :
$$Q = \int_{x_A}^{x_B} dx \sqrt{1+(f')^2} - L$$
(Length fixed!)

$$\tilde{I} = \int_{x_A}^{x_B} dx \, (f+\lambda)\sqrt{1+(f')^2} - \lambda L$$

Let $\tilde{f} = f + \lambda$: extremize $\displaystyle\int \underbrace{\tilde{f}\sqrt{1+(\tilde{f}')^2}}_{F(\tilde{f},\, \tilde{f}')} dx$

indep. of x

you'll $\left\{\begin{array}{l}\text{extremize}\\ \text{minimize}\end{array}\right.$ an identical functional in the problem sets in the context of soap films.

See HW: 3 constants (including λ)

fixed by boundary conditions $R = f(x_A) = f(x_B)$

+ constraint $Q = 0$: 3 equations