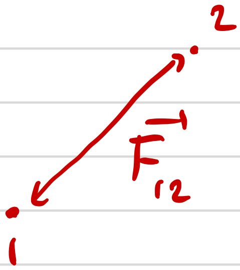
Two-Body Problem

I Two-body problem



$$m_1 \ddot{\vec r}_1 = -\vec\nabla_{\vec r_1} V(|\vec r_1 - \vec r_2|)$$

$$m_2 \ddot{\vec r}_2 = -\vec\nabla_{\vec r_2} V(|\vec r_1 - \vec r_2|)$$

$$\vec\nabla_{\vec r_1} V = \frac{\partial V}{\partial \vec r_1} = \frac{\partial r}{\partial \vec r_1}\frac{dV}{dr} \quad \text{with } \vec r = \vec r_1 - \vec r_2$$

and $$\frac{\partial r}{\partial \vec r_1} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right)\left((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right)^{1/2}$$

$$= \left(\frac{\cancel{2}(x_1-x_2)}{\cancel{2} r}, \frac{y_1-y_2}{r}, \frac{z_1-z_2}{r}\right) = \frac{\vec r_1 - \vec r_2}{r}$$

$$\Rightarrow \quad m_1 \ddot{\vec r}_1 = -V'(r)\frac{\vec r_1 - \vec r_2}{r} \qquad \Rightarrow \quad m_1\ddot{\vec r}_1 + m_2 \ddot{\vec r}_2 = 0$$

$$m_2 \ddot{\vec r}_2 = -V'(r)\frac{\vec r_2 - \vec r_1}{r}$$

Let
$$\boxed{\begin{aligned}\vec R &= \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1+m_2} \\ \vec r &= \vec r_1 - \vec r_2\end{aligned}}$$

center of mass

and $\boxed{(m_1+m_2)\ddot{\vec R} = \vec 0}$

(center of mass ~ free particle)

$$\Leftrightarrow \quad \vec r_1 = \vec R + \frac{m_2}{m_1+m_2}\vec r, \quad \vec r_2 = \vec R - \frac{m_1}{m_1+m_2}\vec r$$

Now $m_1 \ddot{\vec{r}}_1 = m_1 \left(\cancel{\ddot{\vec{R}}}^{\,0} + \frac{m_2}{m_1+m_2} \ddot{\vec{r}} \right) = -\frac{\vec{r}}{r} V'(r)$

$$\Rightarrow \boxed{m \ddot{\vec{r}} = -V'(r)\, \vec{e_r}}$$

$\vec{e_r} = \vec{r}/r$

$m = \frac{m_1 m_2}{m_1+m_2}$ = Reduced Mass

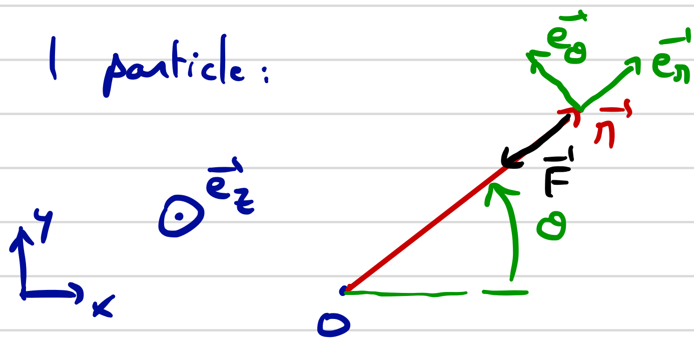
$\Rightarrow$ Effective central problem with particle of mass $m = m_1 m_2/(m_1+m_2)$

Energy: $E = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \underbrace{(m_1+m_2)}_{M} \dot{\vec{R}}^2 + \frac{1}{2} m \dot{\vec{r}}^2$

$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m \dot{\vec{r}}^2 - V(r)$$

II Effective Potential

1 particle:



In spherical coordinates

$$\boxed{\vec{F} = F(r)\, \vec{e_r}}$$ central force $\quad \vec{e_r} = \frac{\vec{r}}{\|\vec{r}\|}$

- Central forces are conservative: $V = -\int^r F(r')\,dr'$
- For central forces the motion takes place on a plane:

$\vec{F} = -\frac{dV}{dr} \vec{e_r} \Rightarrow$ Torque $\vec{T} = \vec{r} \times \vec{F} = \vec{0}$ since $\vec{r} \times \vec{e_r} = \vec{0}$

$\Rightarrow \frac{d\vec{\ell}}{dt} = 0$ with $\vec{\ell} = \vec{r} \times m\vec{v} = \vec{\ell}_0$ constant of motion

$\Rightarrow \vec{r} \in$ plane $\perp$ to $\vec{\ell}_0$ $\Rightarrow$ motion is effectively 2D!

Moreover: $\vec{\ell} = \begin{vmatrix} r \\ 0 \\ 0 \end{vmatrix} \times m \begin{vmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{vmatrix} = m r^2 \dot{\theta}\, \vec{e}_z$

$\Rightarrow$ $\boxed{r^2\dot{\theta} = cst = \frac{\ell_0}{m}}$ $\longrightarrow$ if r small, $\dot{\theta}$ large

Constant of motion (angular momentum)

Equations of motion: (from 2^{nd} law) $m\vec{a} = m \begin{vmatrix} \ddot{r} - r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{vmatrix} = \begin{vmatrix} -dV/dr \\ 0 \end{vmatrix}$

$\Rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d(r^2\dot{\theta})}{dt} = 0$ ✓ we recover $r^2\dot{\theta} = cst$

Let us get rid of $\dot{\theta}$ using $\dot{\theta} = \frac{\ell_0}{mr^2}$

$\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = m\ddot{r} - rm\frac{\ell_0^2}{m^2r^4} = -\frac{dU}{dr}$

$\Leftrightarrow m\ddot{r} = -\frac{dV}{dr} + \frac{\ell_0^2}{mr^3} \equiv -\frac{dU_{eff}}{dr}$

with $\boxed{V_{eff}(r) = V(r) + \frac{\ell_0^2}{2mr^2}}$ Effective potential

we've mapped the problem onto an effective one dimensional system:

$m\ddot{r} = -\frac{dV_{eff}}{dr}$ $\Rightarrow$ formally solvable!

This equation can be integrated once (conservation of energy):

$$\boxed{\frac{1}{2} m \dot{r}^2 + V_{eff}(r) = E = cst}$$

1d effective motion!

$$E = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= \frac{1}{2} m \dot{r}^2 + V_{eff}(r)$$

III Kepler's problem

So the first step is now to plot $V_{eff}(r)$:

Ex: $F(r) = -\frac{K}{r^2}$ (gravitation $-\frac{GMm}{r^2}$, Coulomb $k_e \frac{q_1 q_2}{r^2}$)

$$V(r) = -\frac{K}{r} \Rightarrow V_{eff} = -\frac{K}{r} + \frac{\ell_0^2}{2mr^2}$$

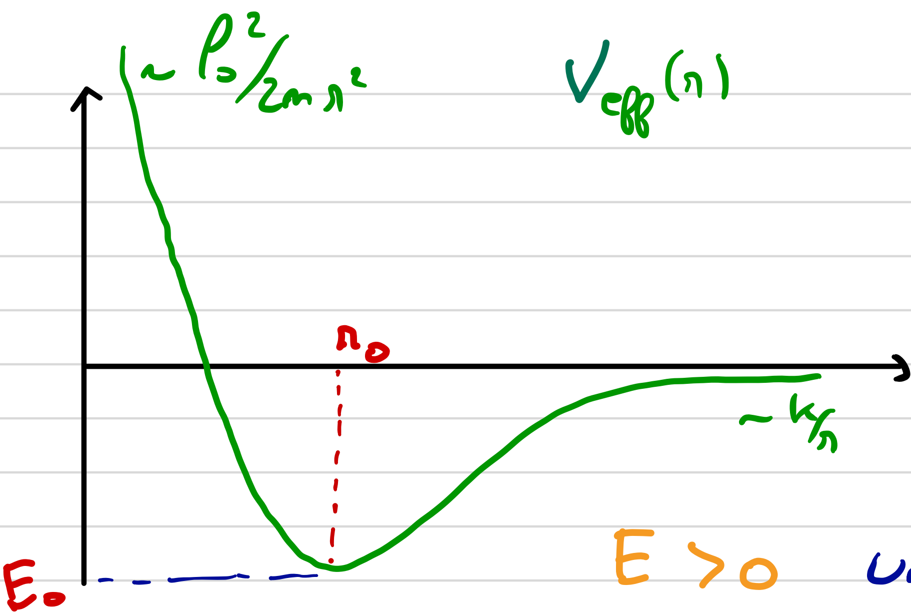
. Note: $E \geq V_{eff}$ for motion to take place

. Circular orbits: $r(t) = r_0$

$$\theta(t) = \theta_0 + \frac{\ell_0}{m r_0^2} t$$

with $V_{eff}'(r_0) = 0 = +\frac{K}{r_0^2} - \frac{\ell_0^2}{m r_0^3} \Rightarrow r_0 = \ell_0^2 / mK$

Period: T_0 : $\frac{\ell_0 T_0}{m r_0^2} = 2\pi \Rightarrow T_0 = 2\pi \frac{m r_0^2}{\sqrt{mKr_0}} = 2\pi \sqrt{\frac{m}{K}} r_0^{3/2}$



$E > 0$ unbounded motion

$E_0 < E < 0$: bounded motion

$E = E_0$: circular motion : $\dot{r} = 0$

Shape of the orbits : $r(t)$, $\theta(t) \Rightarrow r(\theta)$ trajectory (forget time dependence).

$$\dot{r} = \pm\sqrt{\tfrac{2}{m}(E-V) - \ell_0^2/m^2 r^2}$$

$$* \quad \dot{r} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{\ell_0}{mr^2} \Rightarrow \boxed{\theta(r) = \int \frac{\pm \ell_0/mr^2 \; dr}{\sqrt{2/m\,(E-V) - \ell_0^2/m^2r^2}}}$$

Formal solution for any $V(r)$

$$* \quad \ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\theta}\frac{\ell_0}{mr^2}\right) = \dot{\theta}\frac{d}{d\theta}\left(\frac{\ell_0}{mr^2}\frac{dr}{d\theta}\right)$$

$$= \frac{\ell_0}{mr^2}\left[\frac{\ell_0}{mr^2}\frac{d^2r}{d\theta^2} + \frac{dr}{d\theta}\times\left(-\frac{2\ell_0}{mr^3}\frac{dr}{d\theta}\right)\right]$$

$$= \frac{\ell_0^2}{m^2r^4}\frac{d^2r}{d\theta^2} - 2\frac{\ell_0^2}{m^2r^5}\left(\frac{dr}{d\theta}\right)^2$$

so $m\ddot{r} = -V'_{eff} \Rightarrow$

$$\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = \frac{mr^4}{\ell_0^2}F(r)$$

where $F = -V'$

This equation is a bit nasty: non linear and second order!
To solve it, we use the following trick:

$$\boxed{u = \frac{1}{r}} \quad \text{Change of variable}$$

$$\frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$\frac{d^2r}{d\theta^2} = -\frac{d}{d\theta}\left(\frac{1}{u^2}\frac{du}{d\theta}\right) = -\frac{1}{u^2}\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2$$

$$\Rightarrow \frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2 - r = -\frac{1}{u^2}\frac{d^2u}{d\theta^2} \cancel{+\frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2 - 2u \times \frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2} - u^{-1}$$

$$= -\frac{1}{u}\left(\frac{1}{u}\frac{d^2u}{d\theta^2} + 1\right) = \frac{m}{\ell_0^2 u^4} F(1/u)$$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell_0^2 u^2} F(1/u)}$$

"Harmonic oscillator" with driving force $-\frac{m}{\ell_0^2 u^2} F(1/u)$

Kepler problem : $F = -K/r^2 = -Ku^2$

$$\Rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{mK}{\ell_0^2}} \quad \text{easy!}$$

$$\Rightarrow u(\theta) = \frac{mK}{\ell_0^2} + A\cos(\theta - \theta_0)$$

general solution of homogeneous equation

particular solution

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$$r_0 = \ell_0^2/mK$$

Let $A = -\frac{mK}{\ell_0^2}\,\varepsilon$ ← eccentricity $\Rightarrow$ $\boxed{r(\theta) = \frac{r_0}{1-\varepsilon\cos(\theta-\theta_0)}}$

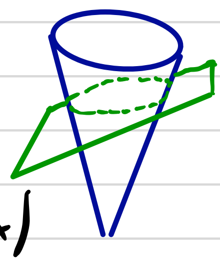
- ε is determined by the system's energy and angular momentum:

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r) = \frac{1}{2}m\frac{\ell_0^2}{m^2 r^4}\left(\frac{dr}{d\theta}\right)^2 - \frac{K}{r} + \frac{\ell_0^2}{2mr^2}$$

$$= \dots = \underbrace{\frac{mK^2}{2\ell_0^2}}_{-E_0}(\varepsilon^2-1) \Rightarrow \boxed{\varepsilon = \sqrt{1+\frac{2E\ell_0^2}{mK^2}}}$$

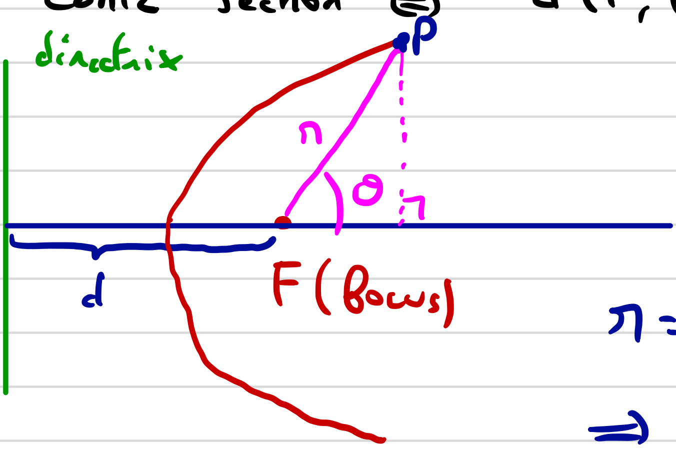
$E>0: \varepsilon>1$
$E<0: \varepsilon<1$
$E=0: \varepsilon=1$
$E=E_{min}: \varepsilon=0$

<u>Orbits</u>: For the Kepler problem, the orbits are conic sections (= hyperbola, parabola, ellipse, circle)



<u>Def.</u> Given a point (focus) and a line (directrix):

$P\in$ conic section $\Leftrightarrow$ $d(P,\text{focus}) = \varepsilon\, d(P,\text{directrix})$



$$d(P,\text{focus}) = r$$

$$d(P,\text{directrix}) = d + r\cos\theta$$

$$r = \varepsilon d + \varepsilon r\cos\theta$$

$$\Rightarrow r = \frac{\varepsilon d}{1-\varepsilon\cos\theta}$$

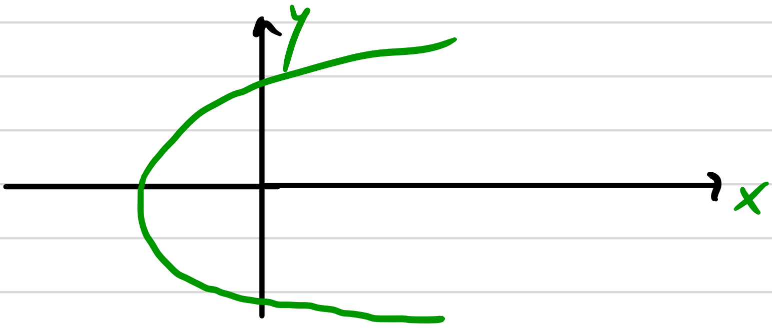
<u>$\varepsilon = 0$</u>: Circle: $d\to\infty$, $\varepsilon\to 0$, $\varepsilon d = r_0$ fixed

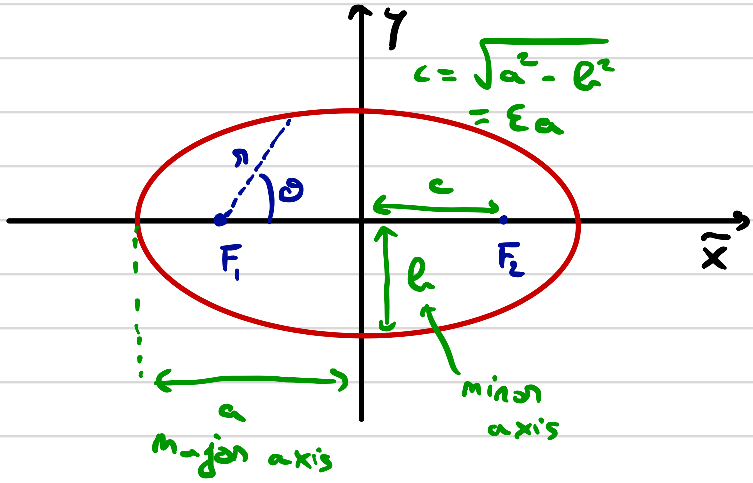
$E = E_{min} = -\frac{mK^2}{2\ell_0^2}$, $r_0 = \ell_0^2/mK$: $\boxed{r(\theta) = r_0}$

circle!

$\varepsilon = 1$: Parabola $\quad (E = 0)$: $\qquad d = r(1-\cos\vartheta) = r - x$

$$\Rightarrow (d+x)^2 = r^2 = x^2 + y^2 \Rightarrow x = \frac{1}{2d}\left(y^2 - d^2\right)$$
$$= x^2 + 2xd + d^2$$





$\varepsilon \neq 0, 1$: Ellipse / Hyperbola
$(0 < \varepsilon < 1)$ $\qquad \varepsilon > 1$
$E < 0$ $\qquad E > 0$

$$\varepsilon d = r - r\varepsilon\cos\vartheta = r - \varepsilon x \Rightarrow \varepsilon^2 (d+x)^2 = r^2 = x^2 + y^2$$
$$= \varepsilon^2 (d^2 + x^2 + 2dx)$$

$$\Leftrightarrow x^2(1-\varepsilon^2) - 2d\varepsilon^2 x + y^2 = \varepsilon^2 d^2$$

$$\Leftrightarrow \left(x - \frac{\varepsilon^2 d}{1-\varepsilon^2}\right)^2 - \frac{\varepsilon^4 d^2}{(1-\varepsilon^2)^2} + \frac{y^2}{1-\varepsilon^2} = \frac{\varepsilon^2 d^2}{1-\varepsilon^2}$$

$$\Rightarrow \left(x - \frac{\varepsilon^2 d}{1-\varepsilon^2}\right)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{\varepsilon^2 d^2}{1-\varepsilon^2}\underbrace{\left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\right)}_{1/(1-\varepsilon^2)} = \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}$$

ellipse

$0 < \varepsilon < 1$: $\quad \dfrac{\tilde{x}^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad \tilde{x} = x - \dfrac{\varepsilon^2 d}{1-\varepsilon^2}$ $\quad$ (closed)

$$a^2 = \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^2}, \quad b^2 = \frac{\varepsilon^2 d^2}{1-\varepsilon^2} \qquad \left(\varepsilon = \sqrt{1 - b^2/a^2}\right)$$

hyperbola

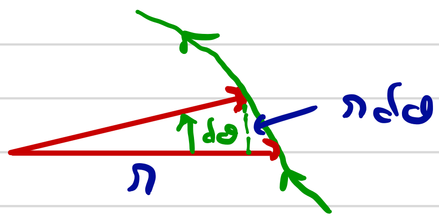
$\varepsilon > 1$: $\quad \dfrac{\tilde{x}^2}{a^2} - \dfrac{y^2}{b^2} = 1, \quad b^2 = \dfrac{\varepsilon^2 d^2}{\varepsilon^2 - 1}$ $\quad$ (open)

IV Kepler's law

① Planets move in elliptical orbits about the sun where the sun at one focus ($M_\odot \gg m_{planets}$: see next chapter)

② The area per unit time swept out by a radius vector from the sun to a planet is constant:

$$r^2\dot{\theta} = cst \quad : \quad dA = \frac{1}{2} r^2 d\theta \qquad \text{so } \frac{dA}{dt} = cst$$



③ The square of a planet's period is proportional to the cube of the major axis of the planet's orbit.

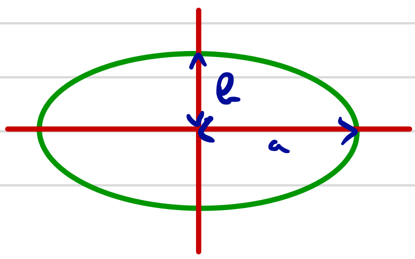
Proof: $\frac{dA}{dt} = \frac{\ell_0}{2m} \Rightarrow \int_0^T dt = T = \frac{2m}{\ell_0} A$ over a period

$A = \pi a b$ for an ellipse

$$= \pi \frac{\varepsilon^2 d^2}{(1-\varepsilon^2)^{3/2}} \quad \text{with } \varepsilon d = \ell_0^2/mk$$

$$= \pi a \; a \times \sqrt{1-\varepsilon^2} \Rightarrow T^2 = \frac{4m^2}{\ell_0^2} A^2 = \frac{4m^2}{\ell_0^2} \pi^2 a^4 (1-\varepsilon^2)$$

and $\varepsilon d = a(1-\varepsilon^2) = \ell_0^2/mk \Rightarrow \boxed{T^2 = \frac{4\pi^2 m}{k} a^3}$



V Integral solution

Let's take a step back and solve the problem a bit more systematically:

$$E = \frac{1}{2} m \dot{r}^2 + V_{eff}(r) \implies \pm t = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{E - V_{eff}(r)}}$$

$$\ell_0 = m r^2 \frac{dr}{d\theta} \implies \theta = \pm \frac{\ell_0}{\sqrt{2m}} \int \frac{dr/r^2}{\sqrt{E - V_{eff}(r)}} \quad \left(\begin{array}{l} + \text{ sign} : \dot{r} > 0 \\ - \text{ sign} \; \dot{r} < 0 \end{array} \right)$$

Focus on bounded case:

$$\Delta\theta = 2 \frac{\ell_0}{\sqrt{2m}} \int_{r_{min}}^{r_{max}} \frac{dr/r^2}{\sqrt{E - V_{eff}(r)}}$$

Orbit closed iff: $n\,\Delta\theta = m\,2\pi \; , \quad n, m \in \mathbb{Z}$

(Exceptional case! Kepler problem, harmonic potential)

Dimensionless variables: $\rho = r/r_0 \; , \; e = E/|E_0| \; , \; \tau = t/T_0$

$$r_0 = \ell_0^2/Km \; , \quad E_0 = -\frac{mK^2}{2\ell_0^2} \; , \quad T_0 = \frac{2\pi \ell_0^3}{mK^2}$$

$$\theta = \pm \int \frac{d\rho/\rho^2}{\sqrt{e - 1/\rho^2 + 2/\rho}} = \pm \arccos\left[(\rho^{-1} - 1) / \sqrt{1+e} \right] + \underset{\pi}{\theta_0}$$

$\tilde{\rho} = \rho^{-1}$

$$\boxed{\rho^{-1} = 1 - \sqrt{1+e}\,\cos\theta}$$ Orbit

$\varepsilon = \sqrt{1+e} \Rightarrow E = E_{min}(1-\varepsilon^2)$

Time Evolution

$$T = \pm \frac{1}{2\pi} \int \frac{d\rho}{\sqrt{e - \rho^{-2} + 2\rho^{-1}}}$$

: can be used to express $\rho(T)$ in parametric form.