Lagrangian Mechanics

I) Action and Euler-Lagrange equation

For a given physical system, we associate a Lagrangian

$$\boxed{L = \underbrace{T}_{\text{kinetic energy}} - \underbrace{V}_{\text{potential energy}}}$$

We then define the action of that system between times t_i and t_f:

$$\boxed{S = \int_{t_i}^{t_f} L(\vec{x}(t), \dot{\vec{x}}(t))\, dt}$$

with $\vec{x}(t_i) = \vec{x}_i$
$\vec{x}(t_f) = \vec{x}_f$

Principle of least action: Physical trajectories are those which minimize the action.

Based on the previous section, we find the Euler-Lagrange Equations:

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0}$$

. Turns out to be crucial for quantum physics, statistical mechanics etc.

. The principle of least action goes way beyond mechanics!

e.g: the standard model of particle physics is formulated in terms of a Lagrangian!

. Even for mechanics only, this formalism is very convenient:

* Since L is a scalar quantity, and is built out off first time derivatives only, it's straightforward to implement changes of coordinate

* if $\frac{\partial L}{\partial x_i} = 0$ then $\frac{\partial L}{\partial \dot{x}_i} = cst$: conserved quantity!

$\boxed{\text{RECIPE}}$

1. Find how many degrees of freedom (# of independent variables need to describe the system) the system has.

$$= \underset{\uparrow\, d=3}{3}\;\underset{\uparrow\, \#\text{ of particles}}{N} - (\#\text{ constraints})$$

2. Choose an appropriate set of generalized coordinates

$$q_i \quad , \qquad i = 1, \dots, M = \#\text{ of d.o.f.}$$

3. Compute $L(q_i, \dot{q}_i)$

4. Write down the Euler-Lagrange equations:

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right) = 0}$$

conjugate momentum

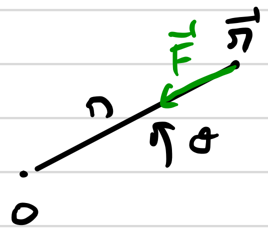
if $\frac{\partial L}{\partial q_i} = 0$, $p_i = \frac{\partial L}{\partial \dot q_i}$ is conserved

II Simple Examples

1 Central Force:

$x = r\cos\theta$
$y = r\sin\theta$

(in 2d, see next chapter)



$$\dot{\vec r}^2 = \dot x^2 + \dot y^2 = \dot r^2 + r^2\dot\theta^2$$

- $L = \frac{1}{2} m\left(\dot r^2 + r^2\dot\theta^2\right) - V(r)$

Now L doesn't depend on θ: $p_\theta = \frac{\partial L}{\partial \dot\theta} = m r^2\dot\theta = cst$

(of course, that's just p_z: angular momentum)

- Equation of motion: $\frac{d}{dt}\frac{\partial L}{\partial \dot r} = \frac{d}{dt}(m\dot r) = \frac{\partial L}{\partial r} = m r\dot\theta^2 - V'(r)$

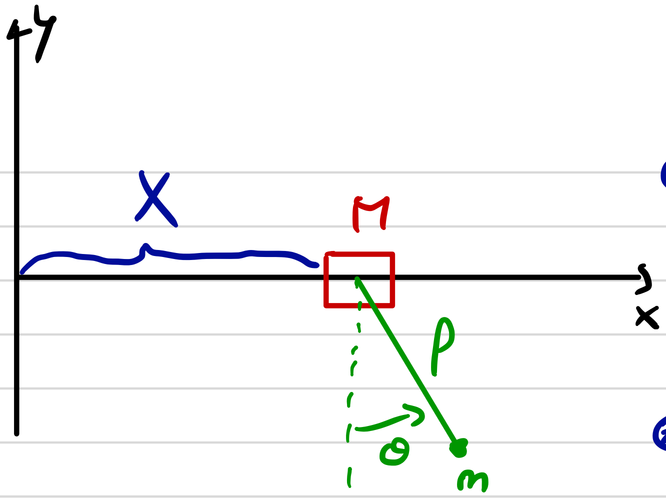
with $p_\theta = p = m r^2\dot\theta = cst$

$$\Rightarrow \quad m\ddot r = m\left(\frac{p}{m r^2}\right)^2 r - V'(r) = \frac{p^2}{m r^3} - V'(r)$$

can be absorbed in $V \to V_{eff}$: next chapter

2 Pendulum attached to a support free to slide horizontally

(without friction)



⊛ 2 d.o.f: X and θ

. M can move in 1d

. For M fixed, simple pendulum.

⊛ $T_M = \frac{M}{2}\dot{X}^2$

⊛ Pendulum mass: $x = X + \ell\sin\theta$

$$y = -\ell\cos\theta$$

$$T_m = \frac{m}{2}\left(\dot{X} + \dot{\theta}\ell\cos\theta\right)^2 + \frac{m}{2}\left(\ell\sin\theta \times \dot{\theta}\right)^2$$

$$= \frac{m}{2}\left(\dot{X}^2 + 2\dot{X}\dot{\theta}\ell\cos\theta + \dot{\theta}^2\ell^2\right)$$

Potential energy: $V = mgy = -mg\ell\cos\theta$ (V_M = cst = irrelevant)

$$\Rightarrow L = \frac{m+M}{2}\dot{X}^2 + \frac{m}{2}\left(2\dot{X}\dot{\theta}\ell\cos\theta + \dot{\theta}^2\ell^2\right) + mg\ell\cos\theta$$

Equations of motion:

X: $\frac{d}{dt}p_X = 0$ with $p_X = \frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} + m\dot{\theta}\ell\cos\theta$

$= \text{cst}$

θ: $\underbrace{\frac{d}{dt}\left(m\dot{X}\ell\cos\theta + m\ell^2\dot{\theta}\right)} = -\cancel{m\dot{X}\dot{\theta}\ell\sin\theta}$

$- mg\ell\sin\theta$

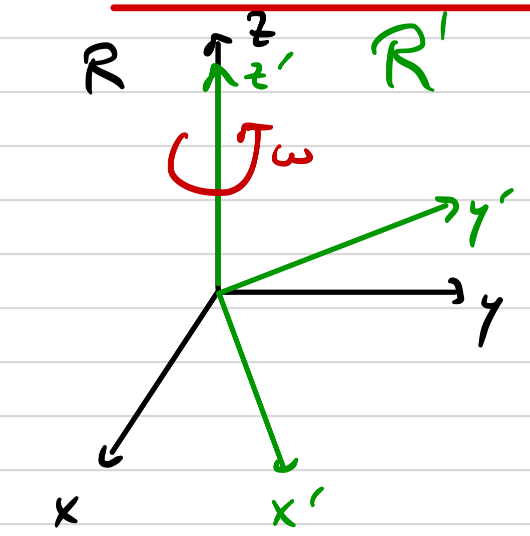
$$\Leftrightarrow \quad m\ddot{X}\ell\cos\theta - m\dot{X}\ell\cancel{\dot{\theta}\sin\theta} + m\ell^2\ddot{\theta}$$

$$\Rightarrow \quad \ddot{x}\cos\theta + \ell\,\ddot{\theta} = -g\sin\theta$$

$$\text{with} \quad \dot{x} = \frac{p_x}{M+m} - \frac{m}{M+m}\,\dot{\theta}\,\ell\cos\theta$$

gives a single equation for θ
($p_x = cst$)

3 Non inertial frame



. Free particle with $L = \frac{1}{2} m \dot{\vec{r}}^2$

$$\vec{r} = \begin{vmatrix} x \\ y \\ z \end{vmatrix}$$

. What's the motion of the particle with respect to R', in rotation with angular velocity $\vec{\omega} = \omega\,\vec{e_z}$ around the z axis?

$$\vec{e_x}' = \cos\omega t\,\vec{e_x} + \sin\omega t\,\vec{e_y}$$
$$\vec{e_y}' = -\sin\omega t\,\vec{e_x} + \cos\omega t\,\vec{e_y}$$

$$\Rightarrow \frac{d}{dt}\vec{e_x}' = \begin{vmatrix} -\omega\sin\omega t \\ \omega\cos\omega t \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ \omega \end{vmatrix} \times \begin{vmatrix} \cos\omega t \\ \sin\omega t \\ 0 \end{vmatrix}$$
$$= \vec{\omega} \times \vec{e_{x'}}$$

Same for $\vec{e_{y'}}$.

$$\Rightarrow \quad \vec{r} = x'\,\vec{e_{x'}} + y'\,\vec{e_{y'}} + z'\,\vec{e_{z'}}$$

$$\dot{\vec{r}} = \underbrace{\dot{x}'\,\vec{e_{x'}} + \dot{y}'\,\vec{e_{y'}} + \dot{z}'\,\vec{e_{z'}}}_{\dot{\vec{r}}'} + \underbrace{\vec{\omega} \times \left(x'\,\vec{e_{x'}} + y'\,\vec{e_{y'}}\right)}_{\vec{\omega} \times \vec{r}'}$$

$$L = \frac{1}{2} m \left(\dot{\vec{r}}' + \vec{\omega} \times \vec{r}'\right)^2 = \frac{1}{2} m \left(\dot{\vec{r}}'^2 + (\vec{\omega} \times \vec{r}')^2 + \dot{\vec{r}}' \cdot (\vec{\omega} \times \vec{r}')\right)$$

$\sum_{\beta,\gamma}$ implicit: Einstein summation convention

Equations of motion: $L = \frac{1}{2} m \sum_\alpha \left(\dot{x}'_\alpha + \varepsilon_{\alpha\beta\gamma}\, \omega_\beta\, x'_\gamma \right)^2$

we have: $$\frac{\partial L}{\partial \dot{x}'_\delta} = m \sum_\alpha \underbrace{\frac{\partial \dot{x}'_\alpha}{\partial \dot{x}'_\delta}}_{\delta_{\alpha\delta}} \left(\dot{x}'_\alpha + \varepsilon_{\alpha\beta\gamma}\, \omega_\beta\, x'_\gamma \right)$$
$$= m \left(\dot{\vec{r}}\,' + \vec{\omega} \times \vec{r}\,' \right)_\delta$$

$$\Longrightarrow \quad \frac{\partial L}{\partial \dot{\vec{r}}\,'} = m \left(\dot{\vec{r}}\,' + \vec{\omega} \times \vec{r}\,' \right)$$

$$\frac{\partial L}{\partial x'_\delta} = m \sum_\alpha \varepsilon_{\alpha\beta\gamma}\, \omega_\beta \underbrace{\frac{\partial x'_\gamma}{\partial x'_\delta}}_{\delta_{\gamma\delta}} \left(\dot{x}'_\alpha + \varepsilon_{\alpha\tilde{\beta}\tilde{\gamma}}\, \omega_{\tilde{\beta}}\, x'_{\tilde{\gamma}} \right)$$

$\alpha \leftrightarrow \gamma = \delta$ $\qquad \sum_{\alpha\beta\tilde{\beta}\tilde{\gamma}}$

$$= -m\, \varepsilon_{\delta\beta\alpha}\, \omega_\beta\, \dot{x}'_\alpha - m\, \varepsilon_{\delta\beta\alpha}\, \omega_\beta\, \varepsilon_{\alpha\tilde{\beta}\tilde{\gamma}}\, \omega_{\tilde{\beta}}\, x'_{\tilde{\gamma}}$$

$\alpha, \beta, \tilde{\gamma}, \tilde{\beta}$ summed over!

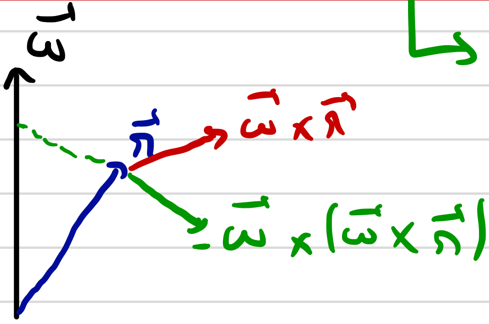
$$= -m \left(\vec{\omega} \times \dot{\vec{r}}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') \right)_\delta$$

So: $$\frac{\partial L}{\partial \vec{r}\,'} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{r}}\,'} \right) \quad \Longleftrightarrow$$

centrifugal force

$$\boxed{m\, \ddot{\vec{r}}\,' = -2m\, \vec{\omega} \times \dot{\vec{r}}\,' - m\, \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')}$$

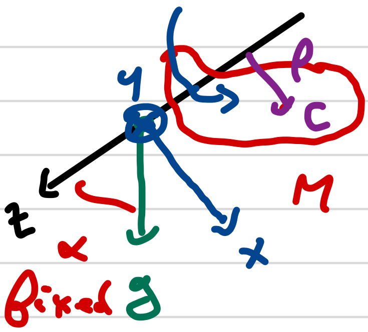
Coriolis "force"



$-\vec{\omega} \times (\vec{\omega} \times \vec{r})$ points outward, away from $\vec{\omega}$

more on that later!
$\|\vec{x}_i - \vec{x}_j\| = \text{fixed}$

4 Rotating Rigid Body



M = Mass of solid
C : center of mass
ρ = dist(C, z axis)
I = moment of inertia w.r.t. z

Solid can rotate around z axis only

$$V = -M\vec{g}\cdot\vec{R} \qquad \vec{R} = \text{center of mass position}$$

$$\vec{R} = \begin{vmatrix} \rho\cos\theta \\ \rho\sin\theta \\ 0 \end{vmatrix} \qquad \vec{g} = \begin{vmatrix} g\sin\alpha \\ 0 \\ g\cos\alpha \end{vmatrix}$$

$$V = -M g \rho \cos\theta \sin\alpha \qquad (\text{min for } \theta = 0 \;\checkmark)$$

$$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} \left(\sum_i m_i \rho_i^2\right) \dot{\theta}^2 = \frac{1}{2} I \dot{\theta}^2$$

moment of inertia, $\rho_i = \text{dist}(i, z)$

$$L_z = \sum_i \rho_i m_i v_i = \sum_i m_i \rho_i^2 \dot{\theta} = I\dot{\theta}$$

$$L = \frac{1}{2} I \dot{\theta}^2 + Mg\rho \cos\theta \sin\alpha$$

$$I\ddot{\theta} = -Mg\rho \sin\alpha \sin\theta$$

$$\boxed{\omega^2 = \frac{Mg\rho\sin\alpha}{I}}$$

Small oscillations, $\theta \ll 1$

III Symmetries and conservation laws

Noether's theorem: Related conservation laws and symmetries

Def: $F(\{q_i\}, \{\dot{q}_i\}, t)$ is a constant of motion (or conserved quantity) if and only if:

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i} \dot{q}_i(t) + \frac{\partial F}{\partial \dot{q}_i} \ddot{q}_i(t) + \frac{\partial F}{\partial t} = 0$$

with $q_i(t)$ satisfying Lagrange's equations.

Claim: If L doesn't depend explicitly on time, then

$$\boxed{H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L}$$

(if we write $p_j := \frac{\partial L}{\partial \dot{q}_j}$, called Hamiltonian, see later)

is a constant of motion $\equiv$ Energy of the system

Time translation invariance $\Rightarrow$ Energy conservation

Proof:

$$\frac{dH}{dt} = \sum_j \cancel{\ddot{q}_j \frac{\partial L}{\partial \dot{q}_j}} + \dot{q}_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} \dot{q}_j - \cancel{\frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j} - \underbrace{\frac{\partial L}{\partial t}}_{=0 \text{ by assumption.}}$$

$= 0$ by Euler-Lagrange

Recall: if $\frac{\partial L}{\partial q_j} = 0$ then $p_j = \frac{\partial L}{\partial \dot{q}_j} = cst$

Noether's theorem: Suppose we have a one-parameter family of maps: $q_i(t) \longrightarrow Q_i(s,t)$ $s \in \mathbb{R}$

such that $Q_i(s=0,t) = q_i(t)$. If this transformation is a continuous symmetry of the system [that is, if $L(Q_i(s,t), \dot{Q}_i(s,t), t)$ doesn't depend on s], then there exists a conserved quantity associated w/ this symmetry.

Proof: $\frac{\partial L}{\partial s} = \sum_i \frac{\partial L}{\partial Q_i} \frac{\partial Q_i}{\partial s} + \frac{\partial L}{\partial \dot{Q}_i} \frac{\partial \dot{Q}_i}{\partial s} = 0$

evaluate at $s=0$: $0 = \left.\frac{\partial L}{\partial s}\right|_{s=0} = \sum_i \frac{\partial L}{\partial q_i} \left.\frac{\partial Q_i}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \dot{q}_i} \left.\frac{\partial \dot{Q}_i}{\partial s}\right|_{s=0}$

E.L. equations

$$= \sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \left.\frac{\partial Q_i}{\partial s}\right|_{s=0} + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}\left(\left.\frac{\partial Q_i}{\partial s}\right|_{s=0}\right)$$

$$= \frac{d}{dt}\left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \left.\frac{\partial Q_i}{\partial s}\right|_{s=0}\right] = 0$$

↳ constant of motion!

Example: Looks abstract? Let's make it concrete!

⊛ Homogeneity of space ⟺ translation invariance of L
⟺ conservation of total linear momentum

$$L = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} - \sum_{j>i} V(|\vec{r}_i - \vec{r}_j|)$$

Has translation symmetry: $\vec{r}_i \to \vec{r}_i + s\vec{n}$, $s \in \mathbb{R}$, any $\vec{n} \in \mathbb{R}^3$

Let's apply Noether's theorem:

Conserved quantity $= \sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot \vec{n} = \sum_i \vec{p}_i \cdot \vec{n}$ conserved $\forall \vec{n}$!

$$\Rightarrow \boxed{\sum_i \vec{p}_i = \vec{cst}}$$

Follow from translation invariance!

$\vec{P}$ total momentum

. Similarly:

Isotropy of space $\Leftrightarrow$ Rotation invariance of L

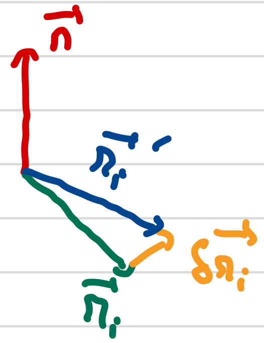
$\Leftrightarrow$ Conservation of total angular momentum

$$\boxed{\sum_i \vec{r}_i \times \vec{p}_i = \vec{cst}}$$

$\vec{L}$ = total angular momentum

Proof: infinitesimal rotation: $\vec{r}_i \longrightarrow \underbrace{\vec{r}_i + \theta\, \vec{n} \times \vec{r}_i}_{\vec{r}_i{}'}$

$\|\vec{r}_i{}'\| = \|\vec{r}_i\|$ to leading order in θ



Conserved quantity: $\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot (\vec{n} \times \vec{r}_i)$

$= \sum_i \vec{n} \cdot (\vec{r}_i \times \vec{p}_i) = \vec{n} \cdot \vec{L}$

angular momentum along $\vec{n}$

invariance under

Homogeneity of time $\Leftrightarrow$ $t \to t + c$: $\partial_t L = 0$

$\Leftrightarrow$ conservation of energy

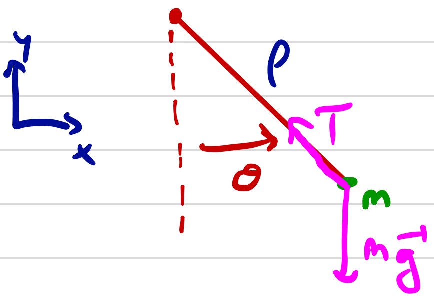
proof: $E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$: $\frac{dE}{dt} = -\frac{\partial L}{\partial t} = 0$

E conserved!

IV Constraints

Often we have to deal with constrained motion.

E.g.: Pendulum



In Newtonian mechanics, we have to introduce a "tension" $\vec{T}$.

Constraint: $\rho^2 = x^2 + y^2$

set $x = \rho \sin\theta$
$y = -\rho\cos\theta$

$$m \begin{vmatrix} \ddot{x} \\ \ddot{y} \end{vmatrix} = \begin{vmatrix} -T\sin\theta = -T\,x/\rho \\ +T\cos\theta - mg \end{vmatrix},$$

or in polar coordinates:

$$m \begin{vmatrix} \ddot{r} - r\dot{\theta}^2 \\ 2\dot{r}\dot{\theta} + r\ddot{\theta} \end{vmatrix} = m \begin{vmatrix} -\rho\dot{\theta}^2 \\ \rho\ddot{\theta} \end{vmatrix} = \begin{vmatrix} -T + mg\cos\theta \\ -mg\sin\theta \end{vmatrix}$$

$$\Rightarrow \left.\begin{aligned} &\ddot{\theta} + g/\rho \sin\theta = 0 \\ &T = mg\cos\theta + m\rho\dot{\theta}^2 \end{aligned}\right\} \text{ can get quickly complicated!}$$

Holonomic constraints : $\boxed{f_\alpha(\{x_i\},t)=0}$ [e.g. $x^2+y^2-\ell^2=0$]

don't involve velocities or inequalities

$\alpha = 1,\dots,3N-n$

$\uparrow$ # constraints

Can be solved using n generalized coordinates $q_1,\dots,q_n$ (n degrees of freedom) [eg: $q=\theta$]

Let's introduce $3N-n$ new variables : λ_α

Lagrange multipliers

ignoring constraints

Lagrangian: $$L = L(\{x_i\}_{i=1,\dots,3N},\{\dot{x}_i\},t) + \sum_\alpha \lambda_\alpha f_\alpha(\{x_i\},t)$$

- treat λ_α like new coordinates (See previous chapter)

$\frac{\partial L}{\partial \lambda_\alpha}=0$ since L doesn't depend on $\dot{\lambda}_\alpha$ $\Rightarrow f_\alpha(\{x_i\},t)=0$

so we recover the constraints.

- Equations for x_i : $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial x_i}$$

Example: Pendulum again:

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - mgy + \lambda \underbrace{(x^2+y^2-\ell^2)}_{\text{constraint}}$$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x^2+y^2=\ell^2$

e.o.m: $m\ddot{x} = 2\lambda x$

$m\ddot{y} = -mg + 2\lambda y$

$\lambda = -\frac{T}{2\ell}$ in Newtonian approach.

So we can easily incorporate constraints in the formalism, but the good news is we don't have to!

We can work directly with generalized coordinates:

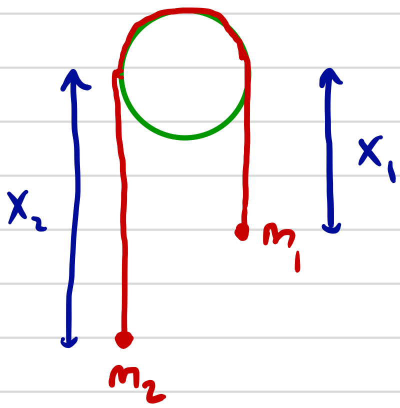
eg: $$L = \underbrace{\frac{1}{2} m \ell^2 \dot{\theta}^2}_{T} + \underbrace{m g \ell \cos\theta}_{-V}$$

$$\Rightarrow \quad m \ell^2 \ddot{\theta} + m g \ell \sin\theta = 0 \quad \checkmark$$

In general, can think of "generalized forces":

$$V_{eff} = -\lambda f(\vec{x}) \quad \Rightarrow \quad \vec{F}_{eff} = +\lambda \vec{\nabla} f$$

Atwood Machine



$$V = -m g x_1 - m g x_2$$

$$L = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + m g x_1 + m g x_2 + \lambda (x_1 + x_2 - C)$$

$$C = x_1 + x_2 = \text{constant}$$

Modified Euler-Lagrange equations: $\quad f = x_1 + x_2 - C$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1 = \frac{\partial L}{\partial x_1} + \lambda \frac{\partial f}{\partial x_1}$$

$$= m_1 g + \lambda \qquad (1)$$

$$m_2 \ddot{x}_2 = m_2 g + \lambda \qquad (2)$$

and $x_1 + x_2 = c \Rightarrow \ddot{x}_1 = -\ddot{x}_2$

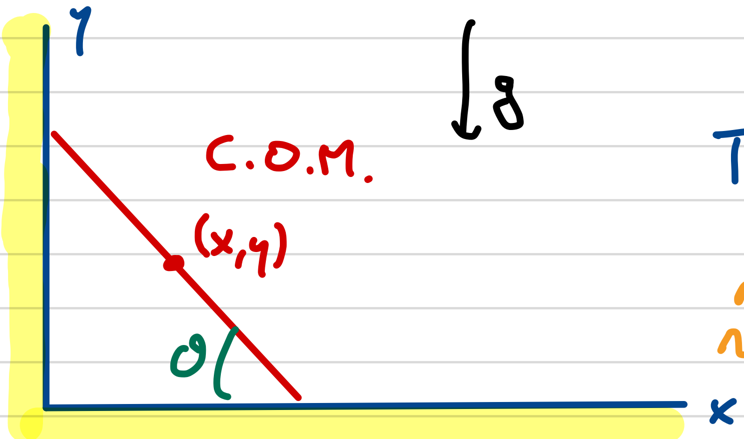
$$\Rightarrow (m_1+m_2)\ddot{x}_1 = (m_1 - m_2) g \qquad (\text{subtract: } (1)-(2))$$

$$\Rightarrow \ddot{x}_1 = \frac{(m_1 - m_2) g}{m_1+m_2} = -\ddot{x}_2 \quad \left(= 0 \text{ if } m_1 = m_2\right)$$

$m_1(1)+(2)m_2$: $\quad 2m_1 m_2 g + (m_1+m_2)\lambda = 0$

$$\Rightarrow \lambda = -\frac{2m_1 m_2 g}{m_1+m_2}$$

Example: Falling ladder



Coordinates: x, y, θ

$$T = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$$

moment of inertia relative to center of mass

$$V = Mgy$$

Constraints: $x = \frac{1}{2}\ell\cos\theta \qquad y = \frac{1}{2}\ell\sin\theta$

contact along wall

$$L = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - Mgy + \lambda_1\left(x - \frac{1}{2}\ell\cos\theta\right)$$
$$+ \lambda_2\left(y - \frac{1}{2}\ell\sin\theta\right)$$

contact along floor

$$M\ddot{x} = \lambda_1$$
$$M\ddot{y} = -Mg + \lambda_2$$
$$I\ddot{\theta} = \frac{\ell}{2}(\lambda_1 \sin\theta - \lambda_2 \cos\theta)$$

$+ \quad x = \frac{\ell}{2}\cos\theta$
$\quad y = \frac{\ell}{2}\sin\theta$

we have: $\dot{x} = -\frac{\ell\dot{\theta}}{2}\sin\theta \,, \quad \ddot{x} = -\frac{\ell\ddot{\theta}}{2}\sin\theta - \frac{\ell\dot{\theta}^2}{2}\cos\theta$

$\dot{y} = \frac{\ell\dot{\theta}}{2}\cos\theta \,, \quad \ddot{y} = \frac{\ell\ddot{\theta}}{2}\cos\theta - \frac{\ell\dot{\theta}^2}{2}\sin\theta$

$$\Rightarrow \lambda_1 = -\frac{\ell M}{2}\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right)$$

$$\lambda_2 = \frac{\ell M}{2}\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) + Mg$$

$$\Rightarrow I\ddot{\theta} = \frac{\ell^2 M}{4}\left(-\ddot{\theta}(\cos^2\theta + \sin^2\theta)\right) - \frac{Mg\ell}{2}\cos\theta$$

$$\Rightarrow \underbrace{\left(I + \frac{\ell^2 M}{4}\right)}_{I'}\ddot{\theta} + \frac{Mg\ell}{2}\cos\theta = 0$$

I' : inertia tensor about end $= I_{\text{com}} + M\left(\frac{\ell}{2}\right)^2$

(see chapter on rigid bodies)

Multiply by $\dot{\theta}$ and integrate:

$$E = \frac{1}{2}I'\dot{\theta}^2 + \frac{Mg\ell}{2}\sin\theta = \frac{Mg\ell}{2}\sin\theta_0 \quad \text{(energy conserved!)}$$

$$\dot{\theta}^2 = \frac{Mg\ell\,(\sin\theta_0 - \sin\theta)}{I'}$$

(can be integrated: "solution by quadrature")

now: $\lambda_1(\theta) = -\frac{\ell M}{2}\left(-\sin\theta\cos\theta\,\frac{Mg\ell}{2I'} + \cos\theta\,\frac{Mg\ell}{I'}(\sin\theta_0 - \sin\theta)\right)$

$$= \frac{\ell^2 M^2 g}{4I'}\cos\theta\,(3\sin\theta - 2\sin\theta_0)$$

$$\lambda_2 = \frac{\ell M}{2}\left(-\cos^2\theta\,\frac{Mg\ell}{2I'} - \sin\theta\,\frac{Mg\ell}{I'}(\sin\theta_0 - \sin\theta)\right) + Mg$$

$$= \frac{\ell^2 M^2 g}{4I'}\left(-\cos^2\theta - 2\sin\theta\sin\theta_0 + 3\sin^2\theta - \sin^2\theta\right) + Mg$$

$$= \frac{\ell^2 M^2 g}{4I'}(3\sin\theta - 2\sin\theta_0)\sin\theta + Mg\underbrace{\left(1 - \frac{\ell^2 M}{4I'}\right)}_{I/I'}$$

Detachment from wall: $\lambda_1 = 0$: $\sin\theta_d = \frac{2}{3}\sin\theta_0$

$$\lambda_2(\theta_d) = Mg\,I/I' > 0$$

time to detachment $T_d = \int_{\theta_0}^{\theta_d} \frac{d\theta}{\dot{\theta}} = \ldots$

Motion after detachment:

$$M\ddot{x} = 0 \quad \Rightarrow \quad \dot{x} = \text{constant} = -\frac{\ell}{2}\sin\theta_d \; \dot{\theta}\Big|_{\theta=\theta_d}$$

$$= -\frac{\ell}{3}\sin\theta_0 \sqrt{\frac{Mg\ell}{I'}\left(\sin\theta_0 - \frac{2}{3}\sin\theta_0\right)} \quad \sim \quad -\sin^{3/2}\theta_0$$

and
$$\begin{cases} M\ddot{y} + Mg = \lambda \\ I\ddot{\theta} = -\frac{1}{2}\ell\lambda\cos\theta \end{cases} \qquad y = \frac{1}{2}\ell\sin\theta$$

$$\ddot{y} = \frac{\ell\ddot{\theta}}{2}\cos\theta - \frac{\ell\dot{\theta}^2}{2}\sin\theta$$

$$\Rightarrow \lambda = Mg + \frac{M\ell}{2}\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right)$$

$$I\ddot{\theta} = -\frac{Mg\ell\cos\theta}{2} - \frac{M\ell^2}{4}\left(\cos^2\theta\,\ddot{\theta} - \cos\theta\sin\theta\,\dot{\theta}^2\right)$$

Integrate:

$$E = \frac{1}{2}\left(I + \frac{M\ell^2}{4}\cos^2\theta\right)\dot{\theta}^2 + \frac{Mg\ell}{2}\sin\theta$$

set by initial conditions at $\theta = \theta_d$, $t = T_d$

again $\theta(t)$ can be solved by quadrature ...

V Motion of a charged particle

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\boxed{L = \tfrac{1}{2} m \dot{\vec{x}}^2 - q(\phi - \dot{\vec{x}}.\vec{A})}$$

charge q

- momentum: $\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + \underbrace{q\vec{A}}_{\text{new term}}$

- Euler Lagrange: $\frac{d}{dt}\left(m\dot{\vec{x}} + q\vec{A}\right) = -q\nabla\vec{\phi} + q\vec{\nabla}(\dot{\vec{x}}.\vec{A})$

$$\Rightarrow m\ddot{\vec{x}} = q\left(\vec{E} + \vec{\nabla}(\dot{\vec{x}}.\vec{A}) - (\dot{\vec{x}}.\vec{\nabla})\vec{A}\right)$$

$$\frac{dA_i}{dt} = \partial_t A_i + \dot{\vec{x}}.\vec{\nabla}A_i$$

and $\vec{\nabla}(\dot{\vec{x}}.\vec{A})\Big|_i = \dot{x}_j \partial_i A_j$

$(\dot{\vec{x}}.\vec{\nabla})\vec{A}\Big|_i = \dot{x}_j \partial_j A_i$

term: $\dot{x}_j \underbrace{(\partial_i A_j - \partial_j A_i)}_{\varepsilon_{ijk} B_k} = (\dot{\vec{x}} \times \vec{B})_i$

$$\Rightarrow \boxed{m\ddot{\vec{x}} = q(\vec{E} + \vec{v} \times \vec{B})}$$

Gauge transformation: $\phi \to \phi + \partial_t \chi$, $\vec{A} \to \vec{A} - \vec{\nabla}\chi$, $\vec{E}, \vec{B}$ unchanged

$$L \to L - q\left(\partial_t \chi + \dot{\vec{r}} \cdot \vec{\nabla} \chi\right) = L - q \frac{d\chi}{dt}$$

$S \to S$: action invariant, equations of motion unchanged.