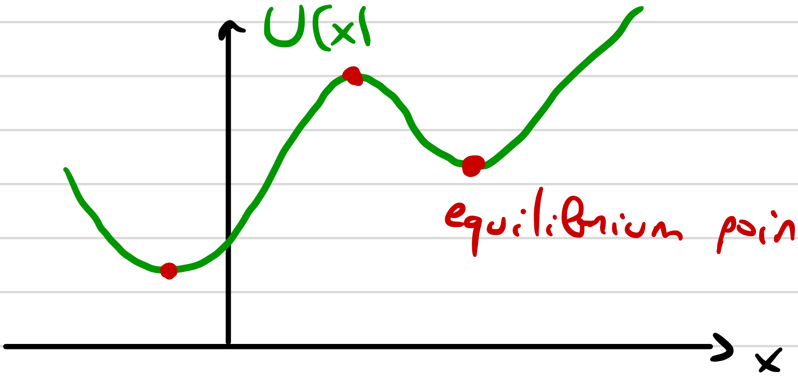
Small Oscillations

I) Single particle

a) Equilibrium, stability and small oscillations

- Focus on 1d system for simplicity, conservative



- Equilibrium points:

$$\left.\frac{dU}{dx}\right|_{x_*} = 0$$

$$(F(x_*) = 0)$$

No motion if we start from these points.

- What about if we go slightly away?

$$x = x_* + \delta x \quad \text{with} \quad \delta x = x - x_* \quad \text{small}$$

Taylor expand:

$$U(x) = \underbrace{U(x_*)}_{\text{Cst}} + \delta x \underbrace{\cancel{U'(x_*)}}_{0 \text{ by definition of } x_*} + \frac{\delta x^2}{2} U''(x_*) + \dots$$

$$\simeq \frac{\delta x^2}{2} U''(x_*) + \text{Cst}$$

$$F = -\frac{dU}{dx} \underset{\uparrow}{=} -\frac{dU}{d\delta x} = -U''(x_*)\,\delta x$$

$d\delta_x = d_x$

$\rightsquigarrow$ looks like a spring if $U''(x_*) > 0$

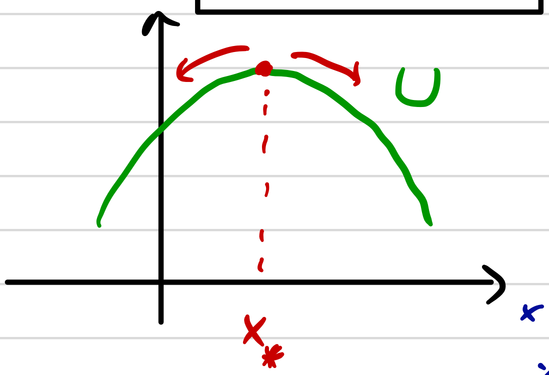
Equation of motion: $\boxed{m\,\ddot{\delta x} + U''(x_*)\,\delta x = 0}$

We need to find two independent solutions. Look for exponentials:

try $e^{\alpha t}$: $m\alpha^2 + U''(x_*) = 0$

$$\Rightarrow \alpha = \pm\sqrt{\frac{-U''(x_*)}{m}}$$

so if $\boxed{U''(x_*) < 0}$:



$$\alpha = \pm\sqrt{\frac{|U''(x_*)|}{m}} = \pm\alpha_*$$

$$\delta x = A e^{-\alpha_* t} + B e^{\alpha_* t}$$

initial conditions: $\delta x_0 = A + B$

$\delta\dot{x}(t=0) = 0 = -\alpha_* A + B\alpha_*$

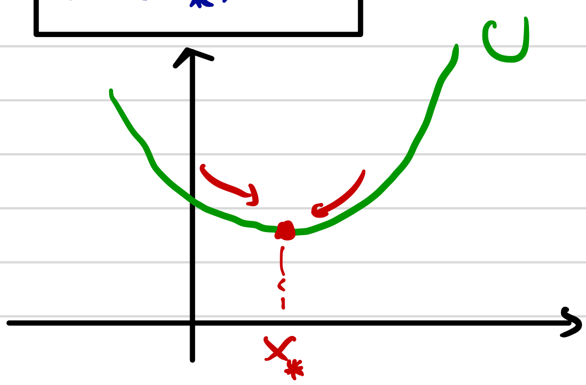
$\Rightarrow A = B$

$$\Rightarrow \delta x(t) = \frac{\delta x_0}{2}\left(e^{\alpha_* t} + e^{-\alpha_* t}\right) = \delta x_0 \cosh(\alpha_* t)$$

$\Rightarrow$ diverges as $t \to \infty$: assumption of small δx breaks down!

$\Rightarrow$ UNSTABLE (moves away from x_*)

if $\boxed{U''(x_*) > 0}$



$$\alpha = \pm i\sqrt{\frac{U''(x_*)}{m}} = \pm i\omega_0$$

$$\boxed{\omega_0^2 = \frac{U''(x_*)}{m}} \quad \left(\delta\ddot{x} + \omega_0^2\,\delta x = 0\right)$$

$\Rightarrow$ $e^{\pm i\omega_0 t}$ solutions or equivalently $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$

$$\Rightarrow \quad \delta x(t) = \delta x_0 \cos(\omega_0 t) \qquad \left(\text{Period } T = 2\pi\sqrt{\frac{m}{U''(x_*)}}\right)$$

$\Rightarrow$ STABLE. Oscillation around x_*.

Approach valid for δx_0 small, otherwise anharmonic terms δx^3, δx^4... become important.

b) Damped Oscillations

- Add a drag / damping force: $\vec{F}_d = -\gamma \dot{x} \vec{e}_x = -\gamma \vec{v}$

$\rightsquigarrow$ non conservative, expect energy loss and damping of oscillations

$$m\ddot{x} = -\gamma \dot{x} - kx \quad \Rightarrow \quad \boxed{\ddot{x} + 2\beta \dot{x} + \omega_0^2 x = 0}$$

$\hookrightarrow \beta = \gamma/2m$

- Look for $e^{\alpha t}$ solutions: $\alpha^2 + 2\beta\alpha + \omega_0^2 = 0$

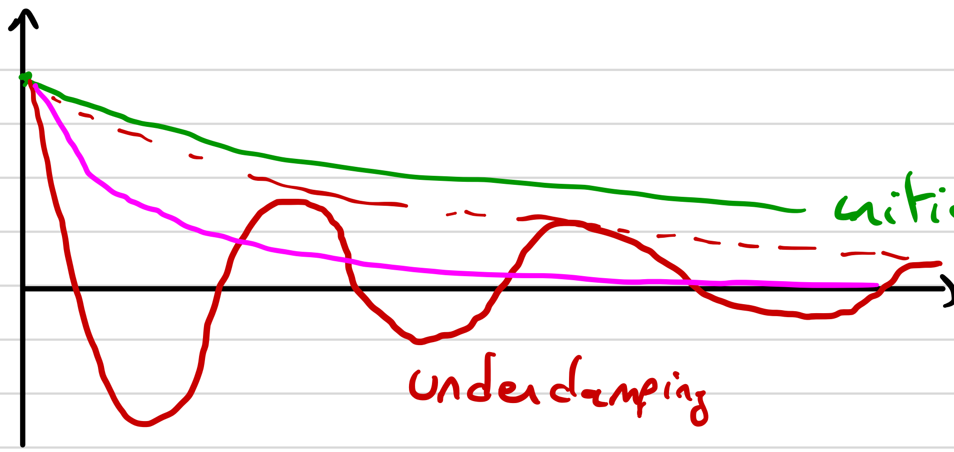
$$\alpha_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

General solution:

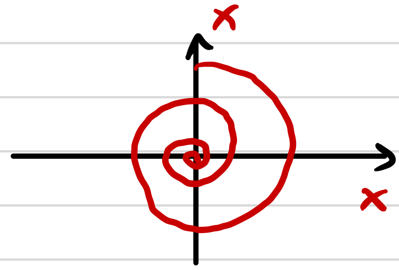
$$x(t) = e^{-\beta t}\left[A e^{\sqrt{\beta^2 - \omega_0^2}\, t} + B e^{-\sqrt{\beta^2 - \omega_0^2}\, t}\right]$$

Three Cases:

- $\omega_0^2 > \beta^2$: under damped motion
- $\beta^2 > \omega_0^2$: overdamped motion
- $\beta^2 = \omega_0^2$: critical case



Phase portrait:



$\omega_0^2 > \beta^2$: $\alpha_\pm = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$

$$x(t) = e^{-\beta t}\left[A e^{i\omega_d t} + B e^{-i\omega_d t}\right]$$

$$= C e^{-\beta t} \cos(\omega_d t + \phi) \qquad \omega_d = \sqrt{\omega_0^2 - \beta^2}$$

$\rightarrow$ damped oscillations

$\omega_0^2 < \beta^2$: $x(t) = e^{-\beta t}\left[A e^{-\omega' t} + B e^{+\omega' t}\right]$

$\omega' = \sqrt{\beta^2 - \omega_0^2} < \beta$: decaying exponentials

$\omega_0^2 = \beta^2$: $x(t) = (A + Bt)\, e^{-\beta t}$

additional solution

c) Periodic driving forces

ex: $F = -Kx - \gamma \dot{x} + F_0 \cos \omega t$

$\beta = \frac{\gamma}{2m}, \quad \omega_0^2 = \frac{k}{m}, \quad A = F_0/m$

$$\Rightarrow \quad m\ddot{x} = F_0 \cos\omega t - kx - \alpha\dot{x} \quad \Rightarrow \quad \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t}$$

Solution: $x(t) = x_c(t) + x_p(t)$

x_p: particular solution

x_c: complementary solution: RHS $= 0$ ("homogeneous eq")

- $$x_c = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

  (see previous section)

  ↳ describes "transient" (decay to 0 as $t \gg \beta^{-1}$)

- Particular solution: try $x_p = D\cos(\omega t - \delta)$

  ↳ same frequency as the drive (Linear eq!)

Trick: write $x_p = \mathrm{Re}\left[\underline{D}\, e^{i\omega t}\right]$, $\quad \underline{D} = D e^{-i\delta} \in \mathbb{C}$

$A\cos\omega t = \mathrm{Re}\left[A e^{i\omega t}\right]$

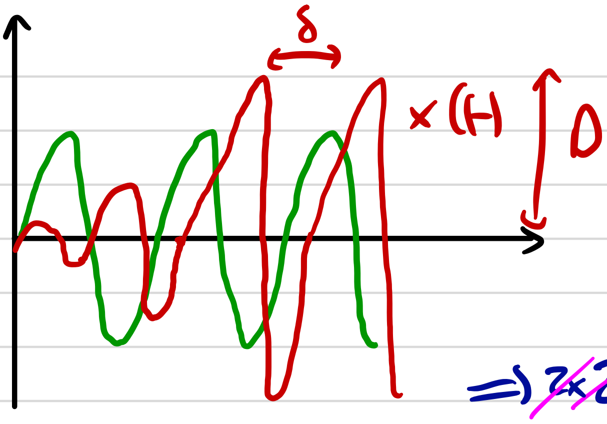
$$\Rightarrow \quad \left[-\omega^2 + 2\beta i\omega + \omega_0^2\right]\underline{D}\, \cancel{e^{i\omega t}} = A\, \cancel{e^{i\omega t}}$$

$$\Rightarrow \quad \underline{D} = \frac{A}{\omega_0^2 - \omega^2 + i2\beta\omega} = \frac{(\omega_0^2-\omega^2)A - i\,2\beta\omega A}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$$

$$\boxed{D = |\underline{D}| = \frac{A}{\sqrt{(\omega^2-\omega_0^2)^2 + 4\beta^2\omega^2}}, \qquad \tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}}$$

↳ "delay", phase difference

$$x(t \gg \beta^{-1}) \simeq x_p(t) \quad \text{(steady-state solution)}$$



Amplitude resonance frequency:

$$\left.\frac{dD}{d\omega}\right|_{\omega_R} = 0$$

$$\Rightarrow \cancel{2}\cancel{\times 2}\,\cancel{\omega_R}\left(\omega_R^2 - \omega_0^2\right) + \cancel{8}\,2\,\beta^2 \cancel{\omega_R} = 0$$

$$\Rightarrow \quad \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

(if $\beta > \omega_0/\sqrt{2}$: no resonance
D decreases monotonically w/ ω)

Quality factor : $Q = \dfrac{\omega_R}{2\beta}$ ($Q \to \infty$ for $\beta \to 0$ no damping)

Principle of superposition

$$\underbrace{\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)}_{\hat{L}} x = A \cos\omega t = F(t)/m$$

$\hat{L}$ = Linear operator: $\hat{L}(x_1 + x_2) = \hat{L}x_1 + \hat{L}x_2$

if $\hat{L}x_1 = F_1$, $\hat{L}x_2 = F_2$ $\Rightarrow$ $\hat{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F_1 + \alpha_2 F_2$

So if $F(t) = \sum_n \alpha_n \cos(\omega_n t - \phi_n)$ (RHS)

the steady-state solution is:

$$x_p(t) = \frac{1}{m}\sum_n \frac{\alpha_n}{\sqrt{(\omega_0^2 - \omega_n^2) + 4\omega_n^2\beta^2}} \cos(\omega_n t - \phi_n - \delta_n)$$

$\hookrightarrow \tan^{-1}\left(\dfrac{2\omega_n\beta}{\omega_0^2 - \omega_n^2}\right)$

So we know the solution for any periodic force $F(t) = F(t+T)$, $T = 2\pi/\omega$ thanks to Fourier Series:

$$F(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t \right)$$

$$a_n = \frac{2}{T} \int_0^T F(t) \cos n\omega t \, dt$$

$$b_n = \frac{2}{T} \int_0^T F(t) \sin n\omega t \, dt$$

d) Response to generic forces: Green's functions

- Suppose we want to solve:

$$\underbrace{\left(\frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2 \right)}_{\hat{L}} x = F(t)/m$$

"generic", not necessarily periodic

Green's function method: solve equation for a pulse:

$F(t) = m\, \delta(t)$: Dirac function: $\int dt\, \delta(t) f(t) = f(0)$

$\delta(t) = 0$ for $t \neq 0$

"$\delta(0) = \infty$"

$$\hat{L} G = \delta(t)$$

(G = Green's function of $\hat{L}$)

Suppose we know how to compute G, then the solution for a general function f is given by the principle of superposition:

$$F(t) = \int_{-\infty}^{+\infty} dt' \, \delta(t-t') \, F(t')$$

"sum" = $\sum\limits_n$; "α_n"

solution for RHS $= \delta(t-t')$

So $$\boxed{x(t) = \int_{-\infty}^{+\infty} dt' \, G(t-t') \, \frac{F(t')}{m}}$$

Check: $$\hat{L}x(t) = \int_{-\infty}^{+\infty} dt' \, \underbrace{\hat{L}G(t-t')}_{\delta(t-t')} \frac{F(t')}{m}$$

$$= \frac{F(t)}{m} \quad ✓$$

How do we compute $G(t)$?

Fourier transform: $$\hat{G}(\omega) = \int_{-\infty}^{+\infty} dt \, e^{i\omega t} G(t)$$

$$G(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{G}(\omega)$$

note that $$\hat{\delta}(\omega) = \int_{-\infty}^{+\infty} dt \, e^{i\omega t} \delta(t) = 1$$

so in Fourier space: $$\left(-\omega^2 - 2\beta i\omega + \omega_0^2\right)\hat{G}(\omega) = 1$$

$\hookrightarrow \frac{d}{dt} \to -i\omega$

$$\Rightarrow \hat{G}(\omega) = -\frac{1}{\omega^2 - \omega_0^2 + 2\beta i\omega} = -\frac{1}{(\omega-\omega_+)(\omega-\omega_-)}$$

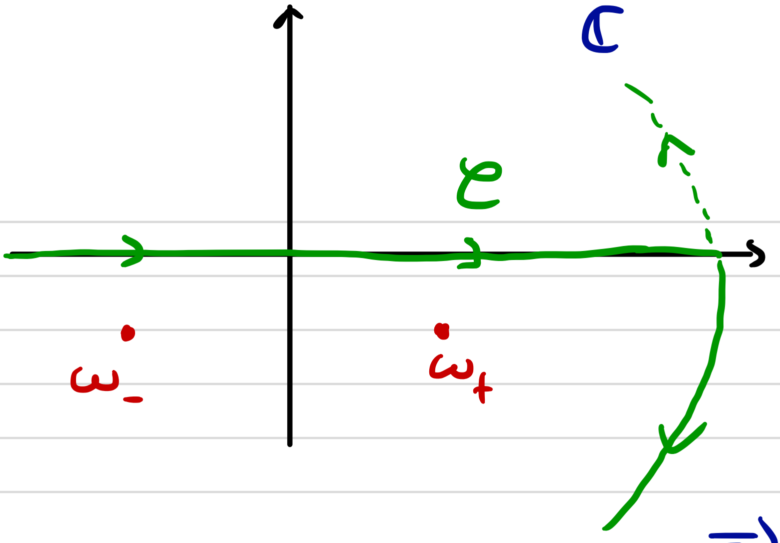
$$\omega_\pm = -i\beta \pm \sqrt{\omega_0^2 - \gamma^2}$$

(same frequencies as damped oscillator!)

To get $G(t)$, we need to Fourier transform back:

$$G(t) = -\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{(\omega-\omega_+)(\omega-\omega_-)}$$

take $\gamma < \omega_0$ for simplicity (underdamped)



if $t<0$, close contour in upper half plane

$$\omega = \omega_R + i\omega_I$$

$$-i\omega t = -i\omega_R t + \underbrace{\omega_I t}_{<0}$$

$\Rightarrow G(t) = 0$ if $t<0$

(no pole)

if $t>0$, use residues: $G(t) = \frac{-2\pi i}{2\pi}\left[\frac{e^{-i\omega_+ t}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_- t}}{\omega_- - \omega_+}\right]$

step function: $\theta(t) = 1$ if $t \geq 0$, $= 0$ otherwise

$$\Rightarrow G(t) = i\,\theta(t) \times e^{-\beta t} \times \frac{1}{2\sqrt{\omega_0^2-\beta^2}}\left[-e^{-i\sqrt{\omega_0^2-\beta^2}\,t} + e^{i\sqrt{\omega_0^2-\beta^2}\,t}\right]$$

$$\Rightarrow \boxed{G(t) = \theta(t)\, e^{-\beta t}\, \frac{\sin\left[\sqrt{\omega_0^2-\beta^2}\; t\right]}{\sqrt{\omega_0^2-\beta^2}}}$$

Note: $G(t-t') = 0$ if $t<t'$ by causality

$x(t) = \int_{-\infty}^{+\infty} dt'\, G(t-t')\, F(t')$ can only depend on $F(t')$ for $t'<t$ ("past") not on $F(t')$ for $t'>t$ ("future")

II Coupled Oscillations

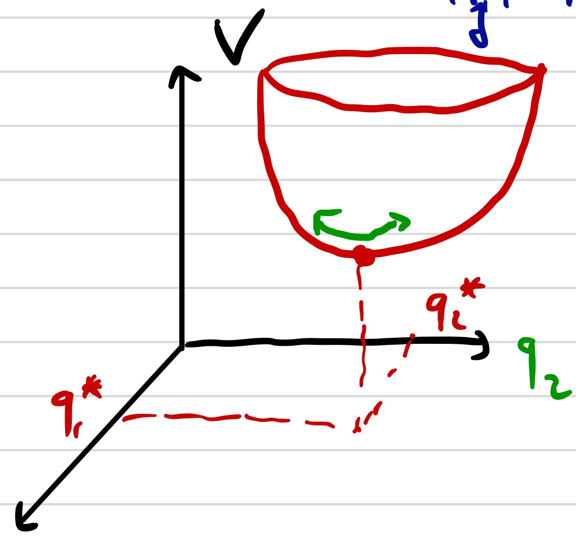
The Lagrangian formalism is very usefull to study small oscillations about equilibrium points of systems with many particles / coordinates.

a) General Approach

$$\mathcal{L} = T(q_i, \dot{q}_i) - V(q_i)$$

$i = 1, \ldots, N$
(generalized coordinates)

Equilibrium: $\left.\frac{\partial V}{\partial q_i}\right|_{\{q_j\} = \{q_j^*\}} = 0$ for $i = 1, \ldots, N$



Small oscillations:

$$q_i = q_i^* + \delta q_i$$

Taylor expand:

$$V = V(\{q_i^*\}) + \sum_j \cancel{\left.\frac{\partial V}{\partial q_j}\right|_{q=q^*}}^{\,0} \delta q_j + \frac{1}{2}\sum_{jk} \left.\frac{\partial^2 V}{\partial q_j \partial q_k}\right|_{q^*} \delta q_j \, \delta q_k + \ldots$$

In general, T is quadratic in $\dot{q}_j$: $T = \sum_{ij} \frac{1}{2} T_{ij} \, \delta\dot{q}_i \, \delta\dot{q}_j$

where $T_{ij} = \left.\dfrac{\partial^2 T}{\partial \dot{q}_i \, \partial \dot{q}_j}\right|_{\substack{\dot{q}_k = 0 \\ q_k = q_k^*}}$

neglect

$$\Rightarrow \boxed{L \simeq \frac{1}{2} \sum_{ij} T_{ij} \, \delta\dot{q}_i \, \delta\dot{q}_j - \frac{1}{2} \sum_{ij} V_{ij} \, \delta q_i \, \delta q_j + \dots}$$

$$V_{ij} = \left.\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right|_{q^*}$$

$\hat{V}$ = N x N matrix with matrix elements V_{ij}

should be positive definite = all eigenvalues should be positive

in order for the equilibrium point to be stable (local minimum vs maximum or saddle point)

Equations of motion: $\dfrac{d}{dt} \dfrac{\partial L}{\partial \delta\dot{q}_i} = \dfrac{\partial L}{\partial \delta q_i}$

$$\Leftrightarrow \sum_j T_{ij} \, \delta\ddot{q}_j = - \sum_j V_{ij} \, \delta q_j \qquad i = 1, \dots, N$$

$$\Leftrightarrow \boxed{\hat{T} \, \delta\ddot{\vec{q}} + \hat{V} \, \delta\vec{q} = \vec{0}} \qquad \delta\vec{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix}$$

Matrix notation

→ Coupled harmonic oscillators

Look for solutions: $\delta\vec{q}(t) = \vec{s}\, e^{i\omega t}$

$$\Rightarrow \quad \hat{T}(i\omega)^2 \vec{s} + \hat{V}\vec{s} = \vec{0}$$

$$\Leftrightarrow \quad (\hat{V} - \omega^2 \hat{T})\,\vec{s} = \vec{0}$$ "Eigensystem"

or in components: $\sum_j (V_{ij} - \omega^2 T_{ij})\, s_j = 0 \quad \forall i$

⊛ In many cases, we'll have $\hat{T}_{ij} = m_i \delta_{ij}$ and the above problem reduces to diagonalizing $\hat{V}$.

⊛ Non-trivial solution:

$$\boxed{\mathrm{Det}(\hat{V} - \omega^2 \hat{T}) = 0} \rightarrow \omega_1^2, \ldots, \omega_N^2$$

N solutions:

Characteristic Equation

(all solutions are real and positive if q^* is a real minimum)

For every ω_k, $\vec{s}_k$: eigenvectors $\Rightarrow$ "Normal Modes"

General solution: $\delta\vec{q} = \sum_k \alpha_k \vec{s}_k\, e^{i\omega_k t} + \beta_k \vec{s}_k^{\,*} e^{-i\omega_k t}$

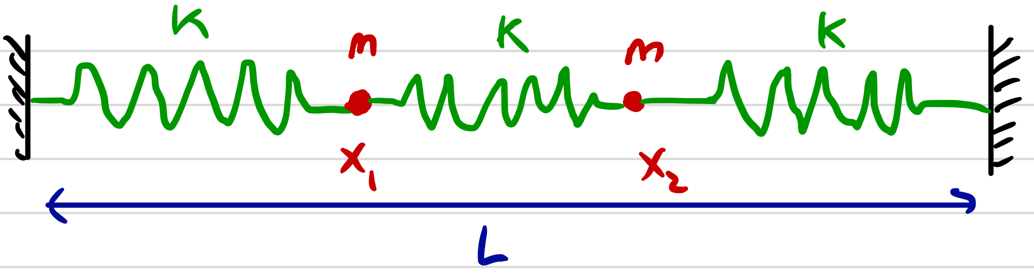
(linear combination)

complex constants, fixed by initial conditions.

$\Rightarrow$ general "recipe" to study coupled oscillations

e) Examples

①



length at rest: $L_0 = L/3$

$$V = \frac{1}{2} k (x_1 - L_0)^2 + \frac{1}{2} k (x_2 - x_1 - L_0)^2 + \frac{1}{2} k (L - x_2 - L_0)^2$$

- Equilibrium: $\frac{\partial V}{\partial x_i} = 0 \implies x_1 = L_0, \quad x_2 = 2L_0$

- small variations: $V = \frac{1}{2} k (\delta x_1)^2 + \frac{1}{2} k (\delta x_2 - \delta x_1)^2 + \frac{1}{2} k (\delta x_2)^2$

$$L = \frac{1}{2} m \left(\delta \dot{x}_1^2 + \delta \dot{x}_2^2 \right) - \frac{1}{2} k \Big[\delta x_1^2 + \delta x_2^2 + \underbrace{\overbrace{(\delta x_1 - \delta x_2)^2}^{\delta x_1^2 + \delta x_2^2 - 2\delta x_1 \delta x_2}}_{\text{coupling}} \Big]$$

$$\hat{T} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \qquad \hat{V} = k \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Characteristic equation: $\begin{vmatrix} 2k - \omega^2 m & -k \\ -k & 2k - \omega^2 m \end{vmatrix} = 0$

$$\iff (2k - \omega^2 m)^2 = k^2 \iff 2k - \omega^2_{\pm} m = \mp k$$

$$\iff \boxed{\omega_+^2 = \frac{3k}{m}, \quad \omega_-^2 = \frac{k}{m}}$$

Normal modes: $\left(\hat{V} - \omega_\pm^2 \hat{T}\right) \begin{pmatrix} s_1^\pm \\ s_2^\pm \end{pmatrix} = 0$

$$\begin{pmatrix} 2k - \frac{3k}{m} m & -k \\ -k & 2k - \frac{3k}{m} m \end{pmatrix} \begin{pmatrix} s_1^+ \\ s_2^+ \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} s_1^+ \\ s_2^+ \end{pmatrix} = 0 \Leftrightarrow s_1^+ = -s_2^+$$

$$\Leftrightarrow \vec{s}^{\,+} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

The motion with frequency ω_+ corresponds to
$\delta x_1(t) = -\delta x_2(t)$

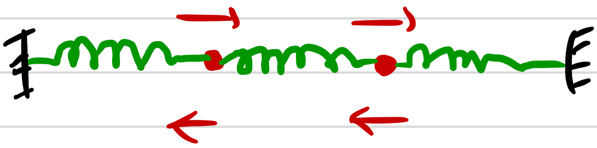
Oscillations out of phase in opposite directions.

$$\begin{pmatrix} 2k - \frac{k}{m} m & -k \\ -k & 2k - \frac{k}{m} m \end{pmatrix} \begin{pmatrix} s_1^- \\ s_2^- \end{pmatrix} = 0$$

$$\Leftrightarrow \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s_1^- \\ s_2^- \end{pmatrix} = 0 \Rightarrow s_1^- = s_2^-$$

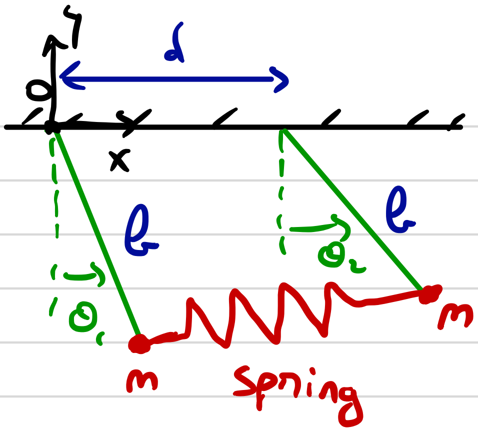
$$\Leftrightarrow \vec{s}^{\,-} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

The motion with frequency ω_- corresponds to $\delta x_1 = +\delta x_2$:



same direction.

2



Assume that spring is unstretched in equilibrium position $\theta_1 = \theta_2 = 0$

$$T = \frac{1}{2} m \ell^2 \dot{\theta}_1^2 + \frac{1}{2} m \ell^2 \dot{\theta}_2^2$$

$$V = \underbrace{-mg\ell\cos\theta_1 - mg\ell\cos\theta_2}_{\text{gravity}} + \frac{1}{2} k (\rho - \rho_0)^2$$

$$(x_1, y_1) = (\ell\sin\theta_1 , -\ell\cos\theta_1)$$

$$(x_2, y_2) = (d + \ell\sin\theta_2 , -\ell\cos\theta_2)$$

$$\rho = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(d + \ell\sin\theta_2 - \ell\sin\theta_1)^2 + \ell^2(\cos\theta_1 - \cos\theta_2)^2}$$

$$\rho_0 = d$$

Small angles approximations: $\sin\theta \simeq \theta$
$\cos\theta \simeq 1 - \theta^2/2$

$$\rho \simeq \sqrt{(d + \ell(\theta_2 - \theta_1))^2 + \cancel{\ell^2\left(\frac{\theta_2^2}{2} - \frac{\theta_1^2}{2}\right)^2}}$$

order θ^4 : drop

$$\simeq d + \ell(\theta_2 - \theta_1)$$

$$\Rightarrow (\rho - \rho_0)^2 = \ell^2 (\theta_2 - \theta_1)^2$$

$$\Rightarrow V = \frac{mg\ell}{2}(\theta_1^2 + \theta_2^2) + \frac{1}{2} k \ell^2 (\theta_2 - \theta_1)^2 + \text{Constant}$$

$$\hat{T} = \begin{pmatrix} m\ell^2 & 0 \\ 0 & m\ell^2 \end{pmatrix}$$

$$\hat{V} = \begin{pmatrix} mg\ell + k\ell^2 & -k\ell^2 \\ -k\ell^2 & mg\ell + k\ell^2 \end{pmatrix}$$

$$\det(\hat{V} - \hat{T}\omega^2) = \begin{vmatrix} mg\ell + k\ell^2 - \omega^2 m\ell^2 & -k\ell^2 \\ -k\ell^2 & mg\ell + k\ell^2 - \omega^2 m\ell^2 \end{vmatrix}$$

$$= 0 \quad \Rightarrow \quad mg\ell + k\ell^2 - \omega^2 m\ell^2 = \pm k\ell^2$$

+ sign: $\omega_+^2 m\ell^2 = mg\ell \quad \Rightarrow \quad \boxed{\omega_+ = \sqrt{g/\ell}}$

− sign: $mg\ell + 2k\ell^2 = \omega_-^2 m\ell^2$

$$\Rightarrow \quad \boxed{\omega_- = \sqrt{\frac{g}{\ell} + \frac{2K}{m}}}$$