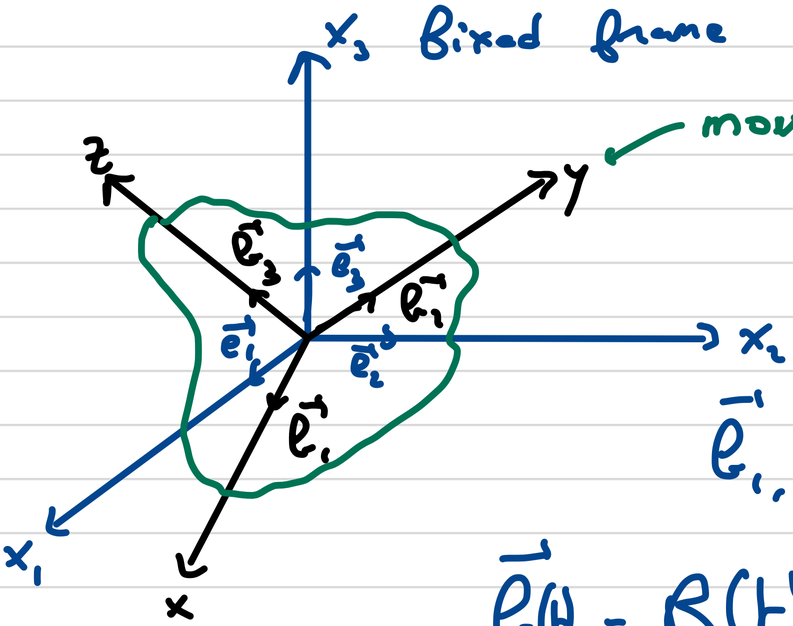
Rigid Bodies

I Kinematics : Rigid bodies : N points s.t. $\|\vec{r}_\alpha - \vec{r}_\beta\| = \text{constant}$

$$M = \sum_\alpha m_\alpha = \int d^3\vec{r}\, \rho(\vec{r}) \quad \text{with} \quad \rho(\vec{r}) = \sum m_\alpha \delta(\vec{x} - \vec{x}_\alpha)$$

6 degrees of freedom: Translations + Rotations
3 3



$$\vec{e}_i \cdot \vec{e}_j = \vec{e}'_i \cdot \vec{e}'_j = \delta_{ij}$$

$\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ all depend on time

$$\vec{e}'_i(t) = R_{ij}(t)\, \vec{e}_j \quad \text{with} \quad R(t) \in SO(3)$$

$$R_{ij}(t) = \vec{e}'_i(t) \cdot \vec{e}_j$$

and $\vec{e}_i = (R^T)_{ij}\, \vec{e}'_j = (\vec{e}_i \cdot \vec{e}'_j)\, \vec{e}'_j$

$$\vec{r}(t) = x_i(t)\, \vec{e}_i = x'_i\, \vec{e}'_i(t) \quad \text{with}$$

point in body — fixed frame — body frame

$$\vec{x} = R\, \vec{x}'(t)$$
$$\vec{x}'(t) = R^T \vec{x}$$

we have $\frac{d\vec{n}}{dt} = \dot{x}_i(t)\vec{e}_i = x_i \dot{\vec{e}}_i = x_i \dot{R}_{ij} \vec{e}_j$

The body frame is changing with time:

$$\dot{\vec{e}}_i = \dot{R}_{ij}\vec{e}_j = \dot{R}_{ij}\underbrace{(R^{-1})_{jk}}_{(R^T)_{jk} = R_{kj}} \vec{e}_k \equiv \omega_{ik}\vec{e}_k$$

$$\omega = \dot{R}R^T \iff \boxed{\dot{R} = \omega R}$$

$R = T\exp\left(\int_0^t \omega\right)$ ↑ time ordered exponential

$$RR^T = \mathbb{1} \Rightarrow \dot{R}R^T + R\dot{R}^T = 0$$

$$\iff \omega + \omega^T = 0 \quad : \omega \text{ antisymmetric}$$

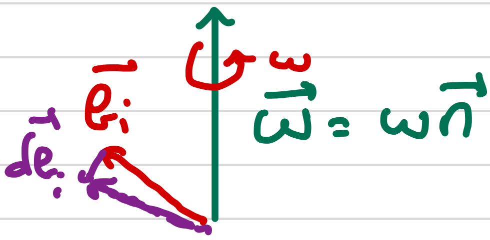
we can parametrize: $\omega_{ij} = \varepsilon_{ijk}\omega_k$ ↳ vector

$$\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad : \quad \dot{\vec{e}}_i = \varepsilon_{ijk}\omega_k \vec{e}_j$$

$$= -\omega_k \underbrace{\varepsilon_{ikj}\vec{e}_j}_{\vec{e}_i \times \vec{e}_k} = \vec{\omega}\times\vec{e}_i$$

$\vec{\omega} = \omega_k \vec{e}_k$: instantaneous angular velocity.

$$\boxed{\dot{\vec{e}}_i = \vec{\omega}\times\vec{e}_i}$$



so $\dot{\vec{n}} = x_i \dot{\vec{e}}_i = \vec{\omega}\times(x_i\vec{e}_i) = \omega\times\vec{n}$

II Inertia tensor : start with rotating body

$$T = \frac{1}{2} \sum_\alpha m_\alpha \vec{r}_\alpha^{\,2} = \frac{1}{2} \sum_\alpha m_\alpha (\vec{\omega} \times \vec{r}_\alpha) . (\vec{\omega} \times \vec{r}_\alpha)$$

labels particles Ren

$$(\vec{a} \times \vec{b})^2 = \varepsilon_{ijk} a_j b_k \varepsilon_{inm} a_n b_m = (\delta_{jn} \delta_{km} - \delta_{jm} \delta_{kn}) \times a_j b_k a_n b_m$$

$$= \vec{a}^2 \vec{b}^2 - (\vec{a} . \vec{b})^2$$

$$\Rightarrow T = \frac{1}{2} \sum_\alpha m_\alpha \left[\omega^2 \vec{r}_\alpha^{\,2} - (\vec{r}_\alpha \vec{\omega})^2 \right]$$

$$= \frac{1}{2} \sum_\alpha m_\alpha \omega_i \left(\delta_{ij} \vec{r}_\alpha^{\,2} - x_i^\alpha x_j^\alpha \right) \omega_j$$

$$= \frac{1}{2} \omega_i I_{ij} \omega_j \quad \text{with :}$$

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \vec{r}_\alpha^{\,2} - x_i^\alpha x_j^\alpha \right)}$$

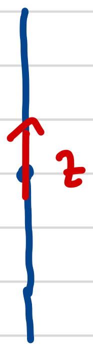
Inertia Tensor

$$I = \int d\vec{r} \, \rho(\vec{r}) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}$$

defined with respect to a point (origin)

- I is time independent, since coordinates $x_1, x_2, x_3 = x, y, z$ are measured with respect to the body frame.
- I is symmetric: we can diagonalize it using an orthogonal matrix: eigenvalues: principle moment of inertia
  eigenvectors: principle axes

Examples Rod (about its center): $\rho = M/\ell$



$$I_z = 0$$

$$I_x = I_y = \int_{-\ell/2}^{\ell/2} x^2 \rho \, dx = \frac{1}{12} M \ell^2$$

Cylinder : $\rho = M/(\pi r_0^2 h)$

$$I_z = h \int_{r<r_0} dx\, dy\, (x^2+y^2)\rho = 2\pi \underbrace{\int_0^{r_0} dr\, r^3}_{r_0^4/4} \frac{M}{\pi r_0^2}$$

$$= \frac{1}{2} M r_0^2$$

$$I_x = I_y = \int dx\, dy\, dz \frac{M}{\pi r_0^2 h} (z^2+y^2)$$

$$= \underbrace{\frac{M}{h} \int_{-h/2}^{h/2} dz\, z^2}_{\frac{1}{2} M h^2} + \frac{M}{\pi r_0^2} \underbrace{\int dx\, dy\, y^2}_{\int_0^{2\pi} d\theta \int_0^{r_0} dr\, r^3 \sin^2\theta = \pi \frac{r_0^4}{4}} \approx \frac{1}{2} M h^2 + \frac{1}{4} M r^2$$

Parallel axis theorem : $I_{\vec{y}}$ = new origin displaced by $\vec{y}$ from center of mass

$$(I_{\vec{y}})_{ij} = (I_{com})_{ij} + M(\vec{y}^2\delta_{ij} - y_i y_j)$$

Proof: $$(I_{\vec{y}})_{ij} = \sum_\alpha m_\alpha \left\{ (\vec{r}_\alpha - \vec{y})^2 \delta_{ij} - (\vec{r}_\alpha - \vec{y})_i (\vec{r}_\alpha - \vec{y})_j \right\}$$

$$= \underbrace{\sum_\alpha m_\alpha \left(\vec{r}_\alpha^{\,2} \delta_{ij} - x_i^\alpha x_j^\alpha \right)}_{(I_{com})_{ij}} + \sum_\alpha m_\alpha \left(\vec{y}^2 \delta_{ij} - y_i y_j \right)$$

$$+ \underbrace{\sum_\alpha m_\alpha \left(-2\vec{y}\cdot\vec{r}_\alpha \delta_{ij} + x_i^\alpha y_j + y_i x_j^\alpha \right)}_{0 \text{ since } \sum_\alpha m_\alpha \vec{r}_\alpha = 0 \text{ if } O = \text{center of mass}}$$

Ex: Inertia tensor of a rod about one of its end:

$$I_x = \frac{1}{12} M \ell^2 + M \left(\frac{\ell}{2}\right)^2 = \frac{1}{3} M \ell^2$$

Energy : $\vec{r}_\alpha(t) = \vec{R} + \vec{\delta}_\alpha(t)$ with com motion $\vec{R}$

$\sum_\alpha m_\alpha \vec{\delta}_\alpha = 0$

$$T = \frac{1}{2} \sum_\alpha m_\alpha \left[\dot{\vec{R}}^2 + \cancel{2\dot{\vec{R}}\cdot\dot{\vec{\delta}}_\alpha}^{\to 0} + \dot{\vec{\delta}}_\alpha^{\,2} \right]$$

$$\boxed{T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \omega_i I_{ij} \omega_j}$$

com motion separates: set $\vec{R} = 0$, $\vec{\delta}_\alpha = \vec{r}_\alpha$

Angular momentum : $\vec{L} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \dot{\vec{r}}_\alpha$

$$= \sum_\alpha m_\alpha \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha) = \sum_\alpha m_\alpha (\vec{r}_\alpha^{\,2}\vec{\omega} - (\vec{\omega}\cdot\vec{r}_\alpha)\vec{r}_\alpha)$$

$$\Rightarrow \boxed{\vec{L} = I\,\vec{\omega}}$$

time dependent!

In body frame : $\vec{L} = L_i\,\vec{e}_i$: $L_i = I_{ij}\,\omega_j$

$\vec{\omega} = \omega_i\,\vec{e}_i$

⚠ $\vec{L}$ and $\vec{\omega}$ point in different directions in general!

III Free motion of a rigid body

Euler's equations : $\vec{F}_{ext} = \vec{0}$, $\vec{R} =$ fixed $=$ origin

we have $\frac{d\vec{L}}{dt} = \vec{0}$ in body frame: $\dot{L}_i\,\vec{e}_i + L_i\,\dot{\vec{e}}_i = \vec{0}$

$$\Leftrightarrow \dot{L}_i\,\vec{e}_i + \underbrace{L_i\,\vec{\omega} \times \vec{e}_i}_{=\vec{L}} = \vec{0} \quad \text{with } \vec{L} = I\vec{\omega}$$

Let's choose $\vec{e}_i$ = principal axes: $L_i = I_i\,\omega_i$

$i = 1,2,3 = x,y,z$

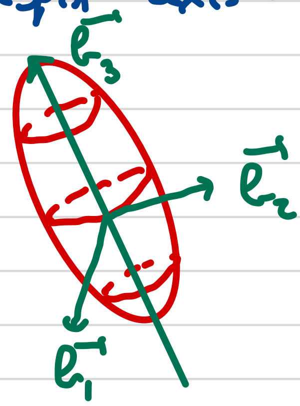
$$\Rightarrow \boxed{I_i\,\dot{\omega}_i + \varepsilon_{ijk}\,\omega_j\,I_k\,\omega_k = 0}$$

Euler's equations

$$I_1\,\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) = 0$$
$$I_2\,\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) = 0$$
$$I_3\,\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) = 0$$

Free tops : sphere : $\vec{\omega}$ = constant (spin axis = const)

Symmetric top : $I_1 = I_2 \neq I_3$



$$I_3 \dot{\omega}_3 = 0$$

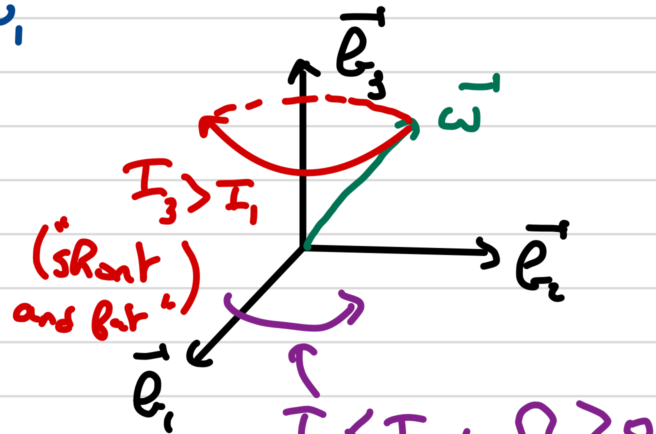
$$\dot{\omega}_1 = \frac{\omega_2 \omega_3}{I_1}(I_1 - I_3)$$

$$\dot{\omega}_2 = -\frac{\omega_1 \omega_3}{I_1}(I_1 - I_3)$$

$\Rightarrow$ ω_3 = const , $\dot{\omega}_1 = \Omega \omega_2$ with $\Omega = \omega_3\left(1 - \frac{I_3}{I_1}\right)$

$\dot{\omega}_2 = -\Omega \omega_1$

we have $\begin{cases} \omega_1 = \omega_0 \sin \Omega t \\ \omega_2 = \omega_0 \cos \Omega t \end{cases}$



$\vec{\omega}$ precesses about $\vec{e}_3$ with frequency Ω! "wobble"

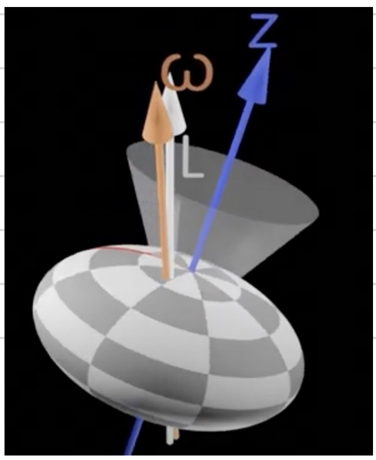
What about in the Lab (fixed) frame?

$\vec{L}$ = const,

and $L_3 = \vec{L} \cdot \vec{e}_3(t)$ = const

angle between $\vec{L}$ and $\vec{e}_3$ fixed

see later (Euler's angles)



stability of asymmetric top

all I_i's different.

$\omega_1 = \Omega$, $\omega_2 = \omega_3 = 0$ solves Euler's equation

$$\vec{\omega} = \begin{vmatrix} \Omega + \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{vmatrix} \; : \quad \begin{aligned} I_1 \dot{\varepsilon}_1 &= 0 \\ I_2 \dot{\varepsilon}_2 &= \Omega \varepsilon_3 (I_3 - I_1) + \mathcal{O}(\varepsilon^2) \\ I_3 \dot{\varepsilon}_3 &= \Omega \varepsilon_2 (I_1 - I_2) \end{aligned}$$

$$\Rightarrow \quad I_2 \ddot{\varepsilon}_2 = \frac{\Omega^2}{I_3} (I_3 - I_1)(I_1 - I_2) \varepsilon_2$$

$$\Rightarrow \quad \ddot{\varepsilon}_2 + \underbrace{\Omega^2 \left(1 - \frac{I_1}{I_3}\right)\left(1 - \frac{I_1}{I_2}\right)} \varepsilon_2 = 0$$

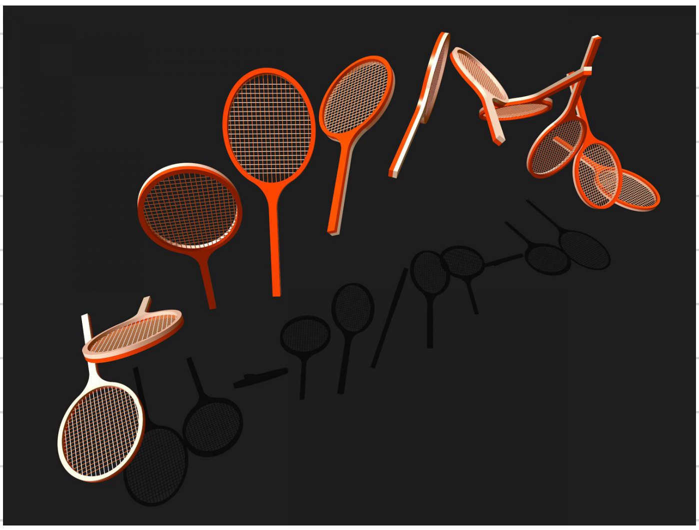
< 0 : unstable : $I_3 < I_1 < I_2$

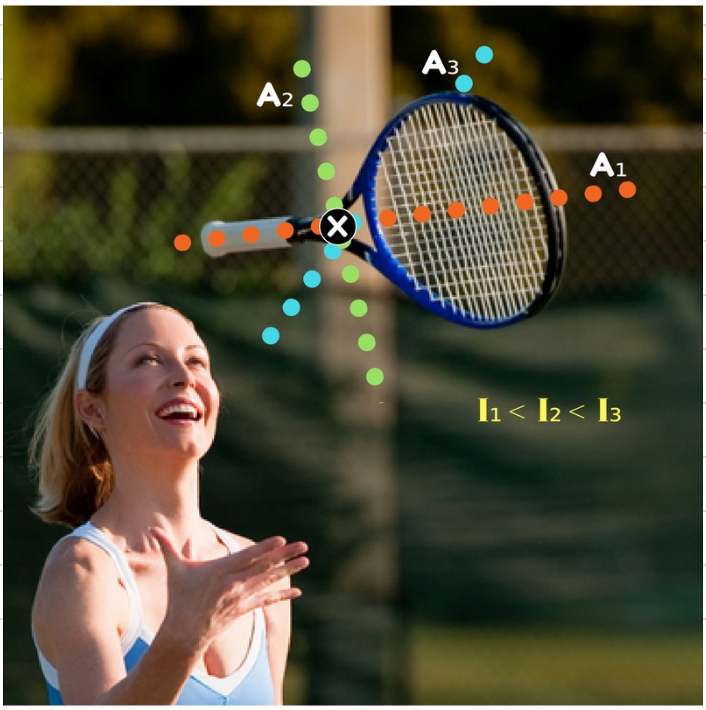
$I_2 < I_1 < I_3$

Stable if I_1 largest or smallest

"Tennis Racket Theorem"

"unstable axis theorem"





Geometric construction

: as before, work in body frame
$\vec{L}$ constant, but $L_i = I_i \omega_i$ evolve

$$\vec{L}^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2 \quad \text{conserved}$$

$$2E = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2$$

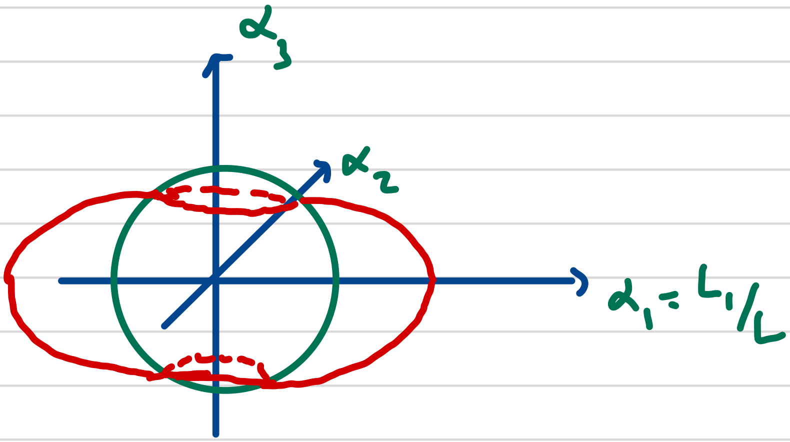
work with $\alpha_i = L_i / L$:

$$\begin{cases} \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 & \text{sphere} \\ \dfrac{\alpha_1^2}{I_1} + \dfrac{\alpha_2^2}{I_2} + \dfrac{\alpha_3^2}{I_3} = \dfrac{2E}{L^2} & \text{ellipsoid} \end{cases}$$

ellipsoid : $I_3 > I_2 > I_1 > 0$

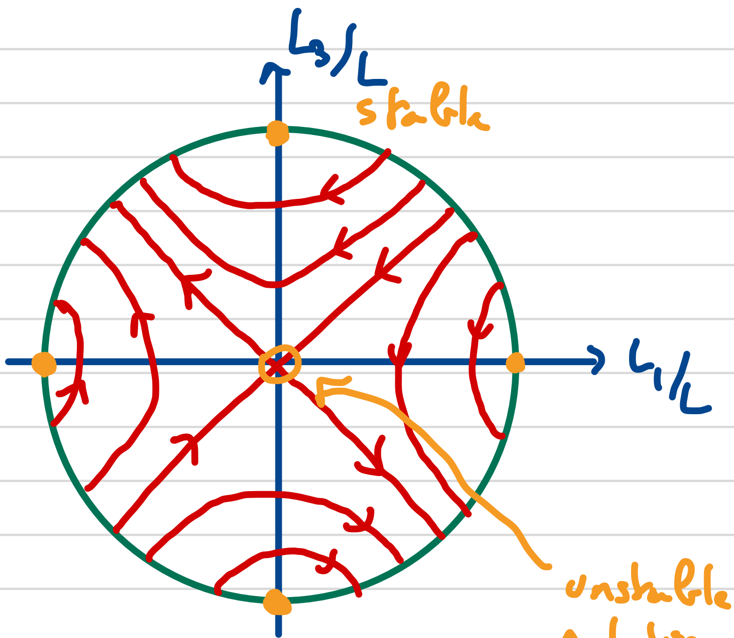
x_1 : major axis

x_3 : minor axis



$\dfrac{1}{I_3} < \dfrac{2E}{L^2} < \dfrac{1}{I_2}$: precession around $\vec{e}_3$

$\dfrac{1}{I_2} < \dfrac{2E}{L^2} < \dfrac{1}{I_3}$: precession around $\vec{e}_1$



$$\frac{2E}{L^2} = \frac{1}{I_2}$$

$$\Rightarrow \quad L_3 = \pm \sqrt{\frac{(I_2/I_1 - 1)}{(1 - I_2/I_3)}}$$

planes intersecting sphere : circles sharing points $(0, \pm 1, 0)$

unstable uniform rotation about $\vec{e}_2$

Motion is periodic in body frame (but not in fixed frame!!

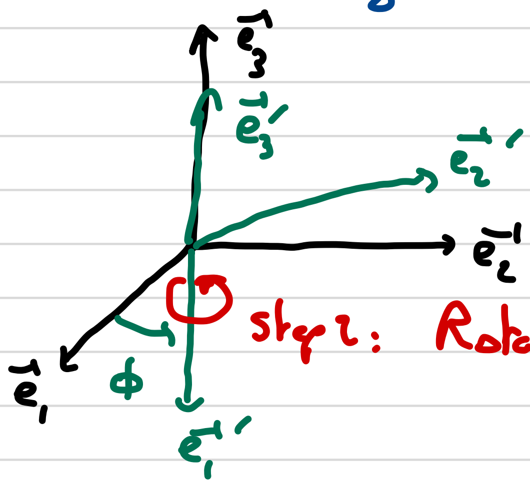
In fixed frame: $T = \frac{1}{2} \vec{L} \cdot \vec{\omega}$: $\vec{\omega}$ has constant component along $\vec{L}$

IV Euler's angles : explicit parametrization of $R(t)$

$$\vec{\mathcal{E}}_i(t) = R(t)_{ij}\, \vec{e}_j$$

$$\{\vec{e}_i\} \xrightarrow{R_3(\phi)} \{\vec{e}_i{}'\} \xrightarrow{R_1(\theta)} \{\vec{e}_i{}''\} \xrightarrow{R_3(\psi)} \{\vec{\mathcal{E}}_i\}$$

① $\vec{e}_i{}' = [R_3(\phi)]_{ij}\, \vec{e}_j$ with $R_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$



step 2: Rotate around $\vec{e}_1{}'$

② $R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$

$\vec{e}_i{}'' = (R_1(\theta))_{ij}\, \vec{e}_j{}'$

③ Rotate around $\vec{e}_3{}''$: $\vec{\mathcal{E}}_i = [R_3(\psi)]_{ij}\, \vec{e}_j{}''$

$$\boxed{R(t) = R_3(\psi)\, R_1(\theta)\, R_3(\phi)}$$

and: $\vec{\omega} = \dot{\phi}\, \vec{e}_3 + \dot{\theta}\, \vec{e}_1{}' + \dot{\psi}\, \vec{e}_3$

expand in body frame: $\vec{e}_3 = \sin\theta \sin\psi\, \vec{e}_1 + \sin\theta\cos\psi\, \vec{e}_2 + \cos\theta\, \vec{e}_3$

$\vec{e}_1{}' = \cos\psi\, \vec{e}_1 - \sin\psi\, \vec{e}_2$

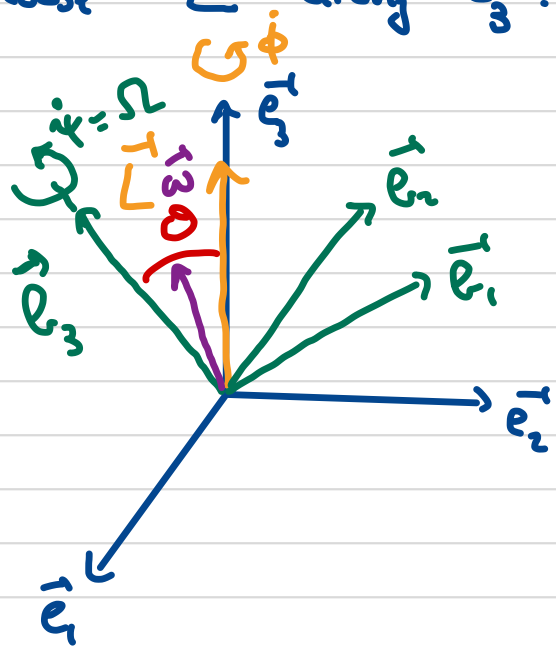
$$\Rightarrow \boxed{\vec{\omega} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\psi} + \dot{\phi}\cos\theta \end{pmatrix}}$$

(in body frame basis)

Ex: free top: $(\omega_1, \omega_2) = \omega_0 (\sin \Omega t, \cos \Omega t)$

$\Omega = \omega_3 (1 - I_3/I_1)$ and constant ω_3 spin

Choose $\vec{L}$ along $\vec{e}_3$: $L_3 = I_3 \omega_3$ conserved



angle between $\vec{e}_3$ and $\vec{e}_3$ = constant = θ

$\dot{\theta} = 0$

$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta = \text{constant}$

$\omega_1 = \dot{\phi}\sin\theta\sin\psi = \omega_0 \sin\Omega t$

$\omega_2 = \dot{\phi}\sin\theta\cos\psi = \omega_0 \cos \Omega t$

$\Rightarrow \psi = \Omega t$ and $\underbrace{\Omega}_{\omega_3 - \frac{I_3}{I_1}\omega_3} + \dot{\phi}\cos\theta = \omega_3$

$$\Rightarrow \dot{\phi} = \frac{I_3 \omega_3}{I_1 \cos\theta}$$

wobble rate = precession

$\dot{\psi}$ = spin

Heavy symmetric top : add gravity, $I_1 = I_2$
fixed point P ($\neq$ COM)

$$L = \frac{1}{2}\tilde{I}_1(\omega_1^2+\omega_2^2) + \frac{1}{2}I_3\omega_3^2 - Mg\ell\cos\theta$$

$$= \frac{1}{2}\tilde{I}_1(\dot\theta^2 + \sin^2\theta\,\dot\phi^2) + \frac{1}{2}I_3(\dot\psi+\cos\theta\,\dot\phi)^2 - Mg\ell\cos\theta$$

$\tilde{I}_1 = I_1 + M\ell^2$; ℓ = dist (P, Center of mass)

- $\frac{\partial L}{\partial \psi} = 0 \Rightarrow p_\psi = (\dot\psi + \cos\theta\,\dot\phi)I_3 = I_3\omega_3 = \text{const.}$

- $\frac{\partial L}{\partial \phi} = 0 \Rightarrow p_\phi = \tilde{I}_1\sin^2\theta\,\dot\phi + \cos\theta\, p_\psi = \text{const.}$

- E conserved: $E = \frac{1}{2}\tilde{I}_1(\dot\theta^2+\sin^2\theta\,\dot\phi^2) + Mg\ell\cos\theta + \underbrace{\frac{1}{2}I_3\omega_3^2}_{\text{const}}$

we have: $$\dot\phi = \frac{p_\phi - \cos\theta\, p_\psi}{\tilde{I}_1\sin^2\theta}$$

$$E = \frac{1}{2}\tilde{I}_1\dot\theta^2 + \underbrace{Mg\ell\cos\theta + \frac{(p_\phi - \cos\theta\, p_\psi)^2}{2\tilde{I}_1\sin^2\theta}}_{V_{\text{eff}}(\theta)} + \text{const}$$