Kepler's problem: Trajectory of a particle in a $1/r$ potential

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Our goal is to study the trajectory of a particle of mass m in a central potential $U(r) = -k/r$ with $k > 0$. Since the torque of the corresponding force is zero, the angular momentum is conserved, and the motion is effectively two-dimensional (in the plane orthogonal to the initial angular momentum). We take the angular momentum to be along the z axis, and use polar coordinates in the xy plane. The conservation of angular momentum gives us that

$$\ell_0 = mr^2\dot{\theta}, \tag{1}$$

is a conserved quantity. From the second law, we have the equation of motion

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2}. \tag{2}$$

Using the conservation law (1) to get rid of $\dot{\theta}$, we find the effective one-dimensional motion $m\ddot{r} = -dU_{\text{eff}}/dr$ with the effective potential:

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell_0^2}{2mr^2}. \tag{3}$$

The energy is given by

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r). \tag{4}$$

We can also express the temporal derivative as

$$\dot{r} = \dot{\theta}\frac{dr}{d\theta} = \frac{\ell_0}{mr^2}\frac{dr}{d\theta}. \tag{5}$$

The simplest way to derive the shape of the trajectory is to use energy conservation and to introduce a new variable u such that $r = 1/u$. We have

$$\dot{r} = \frac{\ell_0 u^2}{m}\frac{dr}{d\theta} = \frac{\ell_0 u^2}{m}\frac{dr}{du}\frac{du}{d\theta} = -\frac{\ell_0}{m}\frac{du}{d\theta}. \tag{6}$$

Let's plug this expression into (4) (using $r = 1/u$ in (3))

$$E = \frac{\ell_0^2}{2m}\left(\frac{du}{d\theta}\right)^2 - ku + \frac{\ell_0^2}{2m}u^2. \tag{7}$$

Now, since energy is conserved along the trajectory, we have

$$\frac{dE}{d\theta} = 0 = \frac{du}{d\theta}\left(\frac{\ell_0^2}{m}\frac{d^2u}{d\theta^2} - k + \frac{\ell_0^2}{m}u\right), \tag{8}$$

which yields

$$\frac{d^2u}{d\theta^2} + u = \frac{km}{\ell_0^2}. \tag{9}$$

This is the equation of a harmonic oscillator with a constant right hand side. The particular solution is that constant, so the general solution reads:

$$u = \frac{km}{\ell_0^2} + A\cos(\theta - \theta_0), \tag{10}$$

where A and θ_0 are integration constants. θ_0 can be chosen by setting a reference angle from which angles are measured, so it is usually set to some convenient choice like $\theta_0 = 0$ or $\theta_0 = \pi$. Going back to $r = 1/u$, we have

$$r(\theta) = \frac{\frac{\ell_0^2}{km}}{1 - \epsilon\cos\theta}, \tag{11}$$

where $\epsilon = -A\ell_0^2/(km)$ and we have chosen $\theta_0 = 0$. This is the equation of a *conic section* in polar coordinates.