Noether's theorem

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Let us prove a more general version of Noether's theorem. Suppose that the Lagrangian is invariant up to a total derivative $L \to L' = L + \epsilon \frac{d\Lambda}{dt}$ (so the action is invariant) under the general infinitesimal transformation

$$t \to t' = t + \epsilon T, \tag{1}$$

$$q_i \to q_i' = q_i + \epsilon Q_i, \tag{2}$$

where $T = \left.\frac{dt'}{d\epsilon}\right|_{\epsilon=0}$ an $Q_i = \left.\frac{dq_i'}{d\epsilon}\right|_{\epsilon=0}$. Then

$$\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) T - \frac{\partial L}{\partial \dot{q}_i} Q_i + \Lambda, \tag{3}$$

is a conserved quantity (using Einstein's summation convention as usual).

Proof: We have

$$L' = L(q_j', \dot{q}_j', t') = L(q_i, \dot{q}_j, t) + \epsilon \frac{d\Lambda}{dt}. \tag{4}$$

Let's Taylor expand:

$$L(q_j + \epsilon Q_j, \dot{q}_j + \epsilon \dot{Q}_j, t + \epsilon T) = L(q_i, \dot{q}_j, t) + \left(\frac{\partial L}{\partial q_i} Q_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Q}_i + \frac{\partial L}{\partial t} T\right)\epsilon + O(\epsilon^2). \tag{5}$$

Now using the invariance of the Lagrangian (4), we find:

$$\frac{\partial L}{\partial q_i} Q_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Q}_i + \frac{\partial L}{\partial t} T = \frac{d\Lambda}{dt}. \tag{6}$$

Now we use Euler-Lagrange $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$ to rewrite the first two terms as

$$\frac{\partial L}{\partial q_i} Q_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) Q_i + \frac{\partial L}{\partial \dot{q}_i}\dot{Q}_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} Q_i\right). \tag{7}$$

Finally, in class, we also showed that (see Chap 3, page 8):

$$\frac{\partial L}{\partial t} = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right). \tag{8}$$

Combining those pieces, we can rewrite eq. (6), we find the conserved charge

$$\frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) T - \frac{\partial L}{\partial \dot{q}_i} Q_i + \Lambda\right] = 0. \tag{9}$$