Hamiltonian Dynamics

Yet another formulation of classical mechanics. Not that useful to solve further complicated problems, but rather reveals the underlying structure of classical mechanics. Also crucial to relate classical and quantum.

I Hamilton's equations

Lagrange: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$ = n 2^{nd}-order differential Equations

Hamilton: n generalized momenta $\boxed{p_i = \frac{\partial L}{\partial \dot{q}_i}}$

$p_i = p_i(q_j, \dot{q}_j, t)$. Place q_i and p_i on equal footing.
$\Rightarrow$ $2n$ first order equations.

Hamiltonian: $\boxed{H = \sum_i p_i \dot{q}_i - L}$ with $\dot{q}_i$ traded for p_i

(Special case of a Legendre transform) (inverting $p_i = p_i(q_j, \dot{q}_j, t)$ $\Rightarrow \dot{q}_i = \dot{q}_i(q_j, p_j, t)$)

Now: $dH = \sum_i dp_i \dot{q}_i + p_i d\dot{q}_i - dL$

$$= \sum_i dp_i \dot{q}_i + \cancel{p_i d\dot{q}_i} - \underbrace{\frac{\partial L}{\partial q_i}}_{\hookrightarrow \dot{p}_i \text{ by Euler-Lagrange}} dq_i - \cancel{\underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{= p_i} d\dot{q}_i} - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \quad dH = -\frac{\partial L}{\partial t}dt + \sum_i \dot{q}_i \, dp_i - \dot{p}_i \, dq_i$$

So $H = H(q_i, p_i, t)$ indeed

(vs $L = L(q_i, \dot{q}_i, t)$)

we also have: $dH = \frac{\partial H}{\partial t}dt + \sum_i \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i$

therefore:

$$\boxed{\begin{aligned} \dot{p}_i &= -\frac{\partial H}{\partial q_i} \\ \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \frac{\partial H}{\partial t} &= -\frac{\partial L}{\partial t} \end{aligned}}$$

Hamilton's equations

$\Rightarrow$ $2n$ 1st order differential equations

Examples:

① Particle in a potential: $L = \frac{1}{2}m\dot{\vec{r}}^2 - V(\vec{r})$

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{r}}} = m\dot{\vec{r}}$$

$$\Rightarrow H = \vec{p}\cdot\dot{\vec{r}} - L = m\dot{\vec{r}}^2 - \frac{1}{2}m\dot{\vec{r}}^2 + V(\vec{r}) = \frac{1}{2}m\dot{\vec{r}}^2 + V(\vec{r})$$

$$\boxed{H = \frac{\vec{p}^2}{2m} + V(\vec{r})}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -\vec{\nabla}V$$

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \vec{p}/m$$

2 Central potential in 2D (Again!)

$$L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - V(r)$$

Momenta: $p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Rightarrow \quad \dot{r} = p_r/m$

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \quad \Rightarrow \quad \dot{\theta} = p_\theta / m r^2$

$$H = p_r \dot{r} + p_\theta \dot{\theta} - L = \frac{p_r^2}{m} + \frac{p_\theta^2}{m r^2} - \frac{1}{2} m \frac{p_r^2}{m^2} - \frac{1}{2} m r^2 \left(\frac{p_\theta^2}{m^2 r^4} \right) + V(r)$$

Hamiltonian:

$$\Rightarrow H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} + V(r)$$

Hamilton's equations:

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{m r^3} - V'(r)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{(conservation of angular momentum!)}$$

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2} \end{aligned}\right\} \text{ we already knew that!}$$

$$\Rightarrow m\ddot{r} = \frac{p_\theta^2}{m r^3} - V'(r) = -\frac{d}{dr} \underbrace{\left(\frac{p_\theta^2}{2 m r^2} + V \right)}_{V_{eff}(r)}$$

3 Charged particle in Electromagnetic field

$$L = \frac{1}{2} m \dot{\vec{x}}^2 - q(\phi - \dot{\vec{x}}.\vec{A})$$

$$\vec{p} = m\dot{\vec{x}} + q\vec{A} \Rightarrow \dot{\vec{x}} = \frac{1}{m}(\vec{p} - q\vec{A})$$

$$\Rightarrow H = \vec{p}.\dot{\vec{x}} - L = \frac{1}{m}\vec{p}.(\vec{p} - q\vec{A}) - \frac{1}{2}\frac{m}{m^2}(\vec{p} - q\vec{A})^2$$

$$+ q\phi - q\frac{\vec{A}}{m}(\vec{p} - q\vec{A})$$

$$\Rightarrow \boxed{H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi}$$

Hamilton's equations:

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = -q\vec{\nabla}\phi + \frac{q}{m}(\vec{p} - q\vec{A}).\vec{\nabla}\vec{A}$$

$$\dot{p}_i = -q\partial_i\phi + \frac{q}{m}(p_j - qA_j)\partial_i A_j$$

and $$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}(\vec{p} - q\vec{A})$$

$$\Rightarrow m\ddot{x}_i + q\partial_t A_i + q\dot{x}_j\partial_j A_i = -q\partial_i\phi + q\dot{x}_j\partial_i A_j$$

$$\Leftrightarrow m\ddot{\vec{x}} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$$

Energy conservation

$$H = H(q_i, p_i, t)$$

Theorem: $\boxed{\frac{dH}{dt} = +\frac{\partial H}{\partial t}}$ so if H doesn't depend explicitly on time then H = energy = conserved quantity!

Proof:
$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_i \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i$$
$$= \frac{\partial H}{\partial t} + \sum_i \underbrace{-\dot{p}_i \dot{q}_i + \dot{q}_i \dot{p}_i}_{0}$$

Hamilton's equations

So if $\partial H/\partial t = 0 \Rightarrow H = E$ = conserved quantity.

Least Action principle revisited

$$S = \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(\vec{q}, \vec{p}) \right) dt \quad :$$

$\delta S = 0$ with δq, δp independent

$$\delta S = \int_{t_1}^{t_2} dt \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i}\delta p_i - \frac{\partial H}{\partial q_i}\delta q_i \right)$$
$$= \int_{t_1}^{t_2} dt \left[\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) + \delta q_i \left(-\dot{p}_i - \frac{\partial H}{\partial q_i} \right) \right]$$
$$+ \cancel{\left[p_i \delta q_i \right]_{t_1}^{t_2}} \; 0$$

$$\delta S = 0 \iff \dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

II Liouville theorem and equation

phase space $\{\vec{q}, \vec{p}\}$

Liouville theorem: Hamiltonian evolution preserves volume in phase space.

Proof: $dV = d\vec{q}\, d\vec{p} \xrightarrow{dt} dV' = d\vec{q}\,' d\vec{p}\,'$

with
$$\vec{q}\,' = \vec{q} + \dot{\vec{q}}\, dt = \vec{q} + \frac{\partial H}{\partial \vec{p}}\, dt$$
$$\vec{p}\,' = \vec{p} - \frac{\partial H}{\partial \vec{q}}\, dt$$

$dV' = |J|\, dV$ with Jacobian $J_{ij} = \begin{pmatrix} \frac{\partial q_i'}{\partial q_j} & \frac{\partial q_i'}{\partial p_j} \\ \frac{\partial p_i'}{\partial q_j} & \frac{\partial p_i'}{\partial p_j} \end{pmatrix}$

$$|J| = \mathrm{Det}\begin{pmatrix} \delta_{ij} + \frac{\partial^2 H}{\partial p_i \partial q_j}\, dt & \frac{\partial^2 H}{\partial p_i \partial p_j}\, dt \\ -\frac{\partial^2 H}{\partial q_i \partial q_j}\, dt & \delta_{ij} - \frac{\partial^2 H}{\partial q_i \partial p_j}\, dt \end{pmatrix}$$

$$= 1 + \sum_i \left(\frac{\partial^2 H}{\partial p_i \partial q_i} - \frac{\partial^2 H}{\partial q_i \partial p_i} \right) dt + \mathcal{O}(dt^2) = 1 + \mathcal{O}(dt^2)$$

where we've used $\det(\mathbb{I}+\varepsilon M) = 1+\varepsilon \operatorname{Tr} M + \mathcal{O}(\varepsilon^2)$

$\Rightarrow \quad dV = dV'$

dN particles in volume dV : $\rho = \frac{dN}{d\Gamma}$: $\frac{d\rho}{dt} = 0$

$\rho(q_i, p_i, t)$ = phase space distribution

$$N = \int d\vec{p}\, d\vec{q}\, \rho .$$

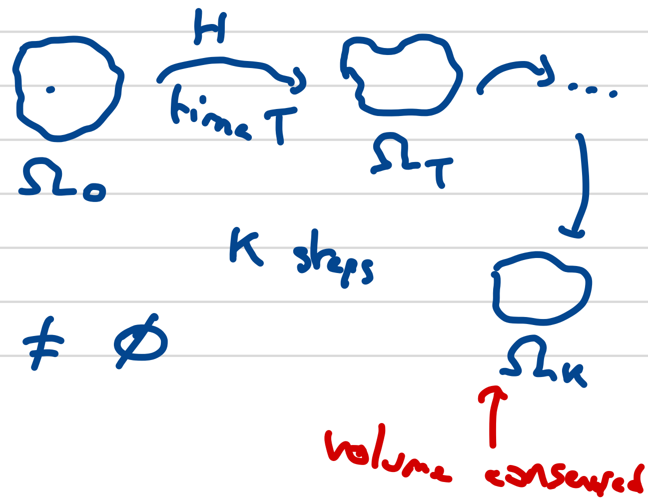
We have : $\dot{\rho} = 0 = \partial_t \rho + \frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i$

$$= \partial_t \rho + \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial \rho}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$\Rightarrow \boxed{\partial_t \rho = \frac{\partial H}{\partial q_i}\frac{\partial \rho}{\partial p_i} - \frac{\partial \rho}{\partial q_i}\frac{\partial H}{\partial p_i} = \{H, \rho\}}$$

Poisson Bracket
next section

Poincaré Recurrence theorem :



$\bigcup_K \Omega_K < \infty$ for finite phase space

$\exists K, K'$ such that $\Omega_K \cap \Omega_{K'} \neq \emptyset$

Time evolve back: $\Omega_0 \cap \Omega_{k'-k} \neq \emptyset$!

System goes back arbitrarily close to P (neighborhood Ω_0 arbitrary!!) in finite time!

III Poisson brackets and canonical transformations

Poisson Brakets: Given two functions $f(q_i, p_i, t)$; $g(q_i, p_i, t)$

$$\boxed{\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)}$$

It satisfies:

$$\{f, f\} = 0$$

$$\{f, g\} = -\{g, f\}$$

$$\{f, C\} = 0 \quad \text{if } C \text{ is a constant}$$

$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$$

$$\{f_1 f_2, g\} = f_1 \{f_2, g\} + \{f_2, g\} f_1$$

$$\frac{\partial}{\partial t} \{f, g\} = \left\{\frac{\partial f}{\partial t}, g\right\} + \left\{g, \frac{\partial f}{\partial t}\right\}$$

(All of the above are very easy to prove)

Jacobi Identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

(Proof requires patience and a large cup of coffee!
Brute force expand and check that it works!)

Why? Say we have a function $f(q_i(t), p_i(t), t)$

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t}$$

$$= \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i}$$

Hamilton's Equations

so $\displaystyle \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$.

If f doesn't depend explicitly on time: $\displaystyle \frac{\partial f}{\partial t} = 0$

$$\Rightarrow \boxed{\frac{df}{dt} = \{f, H\}} \quad \text{for any } f(q_i, p_i)$$

In particular:

$$\boxed{\begin{aligned} \dot{q}_i &= \{q_i, H\} \\ \dot{p}_i &= \{p_i, H\} \end{aligned}}$$

Hamilton's Equations

$\{f, H\} = 0$, then f is a constant of motion (= conserved quantity)

$\{H, H\} = 0$ so if $\dfrac{\partial H}{\partial t} = 0$, H is a conserved quantity = energy!

Ex: Angular momentum

$$\ell_i = \varepsilon_{ijk}\, x_j\, p_k$$

$$\{\ell_1, \ell_2\} = \ell_3 \quad , \quad \{\vec{\ell}^{\,2}, \ell_i\} = 0$$

Fundamental Poisson Brackets :

$$\{q_i, q_j\} = 0 \qquad \text{since} \quad \frac{\partial q_i}{\partial p_j} = 0$$

$$\{p_i, p_j\} = 0 \qquad \text{since} \quad \frac{\partial p_i}{\partial q_j} = 0$$

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} = \sum_k \delta_{ik}\delta_{jk} = \delta_{ij}$$

$$\Rightarrow \quad \boxed{\{q_i, p_j\} = \delta_{ij}}$$

Canonical transformations :

- In Lagrangian formalism, the Euler-Lagrange equations are invariant under any change of coordinates:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = \frac{\partial L}{\partial Q_i} \quad \text{for any } Q_i = Q_i(q_j)$$

Proof: $\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \partial_t Q_i \, , \quad \dot{q}_i = \frac{\partial q_i}{\partial Q_j}\dot{Q}_j + \partial_t q_j$

$$\frac{\partial L}{\partial Q_i} = \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial}{\partial Q_i}\left(\frac{\partial q_j}{\partial Q_k}\dot{Q}_k + \partial_t q_j\right)$$

$$\frac{\partial L}{\partial \dot{Q}_i} = \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial \dot{Q}_i} = \frac{\partial L}{\partial \dot{q}_j}\frac{\partial q_j}{\partial Q_i}$$

$$\dot{q}_j = \partial_t q_j + \frac{\partial q_j}{\partial Q_k}\dot{Q}_k$$

so
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\frac{\partial q_j}{\partial Q_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial}{\partial Q_i}\left(\partial_t q_j + \frac{\partial q_j}{\partial Q_k}\dot{Q}_k\right)$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) - \frac{\partial L}{\partial Q_i} = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right)\frac{\partial q_j}{\partial Q_i} = 0 \quad \checkmark$$

invertible matrix

. Now imagine a transformation mixing q's and p's

$$Q_i = Q_i(q_j, p_j)$$
$$P_i = P_i(q_j, p_j)$$

Do we have: $\dot{Q}_i = \partial H/\partial P_i$?

$\dot{P}_i = -\partial H/\partial Q_i$

YES if Q and P are canonical:

$$\boxed{\begin{aligned} \{P_i, P_j\} &= 0 \\ \{Q_i, Q_j\} &= 0 \\ \{Q_i, P_j\} &= \delta_{ij} \end{aligned}}$$

. To show this, it turns out to be very usefull to use the symplectic structure of Hamiltonian dynamics:

Let $\vec{\phi} = (\vec{q}, \vec{p})$ 2N - dim vectors

then $\dot{\vec{\phi}} = J \frac{\partial H}{\partial \vec{\phi}}$ with $J = \begin{pmatrix} 0 & +\mathbb{1}_{N\times N} \\ -\mathbb{1}_{N\times N} & 0 \end{pmatrix}$

New coordinates: $\vec{\varphi} = \vec{\varphi}(\vec{\phi})$

$$\dot{\varphi}_i = \frac{\partial \varphi_i}{\partial \phi_j} \dot{\phi}_j = \frac{\partial \varphi_i}{\partial \phi_j} J_{ik} \underbrace{\frac{\partial H}{\partial \phi_k}}_{\frac{\partial H}{\partial \varphi_n} \frac{\partial \varphi_n}{\partial \phi_k}} = \left(I J I^T \right) \frac{\partial H}{\partial \vec{\phi}} \Big|_i$$

$I = \frac{\partial \vec{\varphi}}{\partial \vec{\phi}}$ Jacobian matrix

Canonical transformation iif : $\boxed{I J I^T = J}$

I symplectic matrix

Now: ① Poisson bracket invariant under canonical transformation

Proof: $\{f, g\} = \frac{\partial f}{\partial \vec{\phi}} J \frac{\partial g}{\partial \vec{\phi}}$ so if $\vec{\phi} \to \vec{\varphi}(\vec{\phi})$

$$\frac{\partial f}{\partial \phi_i} = \frac{\partial f}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial \phi_i}$$

$$\Rightarrow \{f, g\} = \frac{\partial f}{\partial \vec{\varphi}} \underbrace{I J I^T}_{J \text{ for a canonical transformation}} \frac{\partial g}{\partial \vec{\varphi}}$$

2 if $\vec{\phi}_{(\vec{q},\vec{p})} \longrightarrow \vec{\varphi}_{(\vec{Q},\vec{P})}$ preserves Poisson Brackets

$\{Q_i, P_j\} = \delta_{ij}$ etc

then it is canonical

Proof: Jacobian: $I_{ij} = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix}$

$$\Rightarrow \quad I J I^T = \begin{pmatrix} \{Q_i, Q_j\} & \{Q_i, P_j\} \\ \{P_i, Q_j\} & \{P_i, P_j\} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \delta_{ij} \\ -\delta_{ij} & 0 \end{pmatrix} = J \quad ✓$$

Generators : $(q_i, p_i) \longrightarrow (Q_i = q_i + \varepsilon F_i,\ P_i = p_i + \varepsilon G_i)$

$$I_{ij} = \begin{pmatrix} \delta_{ij} + \varepsilon \frac{\partial F_i}{\partial q_j} & \varepsilon \frac{\partial F_i}{\partial p_j} \\ \varepsilon \frac{\partial G_i}{\partial q_j} & \delta_{ij} + \varepsilon \frac{\partial G_i}{\partial p_j} \end{pmatrix}$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & C \\ B & D \end{pmatrix}}_{\begin{pmatrix} B & D \\ -A & -C \end{pmatrix}} = \begin{pmatrix} AB - BA & AD - BC \\ CB - DA & CD - DC \end{pmatrix}$$

I_{ij} symplectic iff $AD = \mathbb{1}$ to $O(\varepsilon)$

$$\Leftrightarrow \frac{\partial F_i}{\partial q_j} = -\frac{\partial G_i}{\partial p_j}$$ satisfied if $F_i = \frac{\partial \Phi}{\partial p_i}$

$G_i = -\frac{\partial \Phi}{\partial q_i}$

$\Phi(\vec{q}, \vec{p})$ = generator of canonical transformation

$$\boxed{\begin{aligned} \frac{dQ_i}{d\varepsilon} &= \frac{\partial \Phi}{\partial p_i} \\ \frac{dP_i}{d\varepsilon} &= -\frac{\partial \Phi}{\partial q_i} \end{aligned}}$$

"Hamiltonian Flow" in phase space!

"time" = ε here

Φ = Hamiltonian

Ex: $\Phi = p_k$: $q_i \to Q_i = q_i + \varepsilon\, \delta_{ik}$

↳ p_k generates translations of q_k.

Noether's theorem again: generator Φ

$$\delta H = \frac{\partial H}{\partial q_i} \delta q_i + \frac{\partial H}{\partial p_i} \delta p_i = \varepsilon \left(\frac{\partial H}{\partial q_i} \frac{\partial \Phi}{\partial p_i} - \frac{\partial H}{\partial p_i} \frac{\partial \Phi}{\partial q_i} \right)$$

$$= \varepsilon \{ H, \Phi \}$$

Φ generates a symmetry if $\delta H = 0 \Leftrightarrow \{H, \Phi\} = 0$

$\Leftrightarrow \dot{\Phi} = \{\Phi, H\} = 0$

goes both ways! Φ symmetry $\Rightarrow$ Φ conserved

Φ conserved $\Rightarrow$ Φ generates a symmetry

IV Action-Angle variables and integrable systems

Action-Angle variables

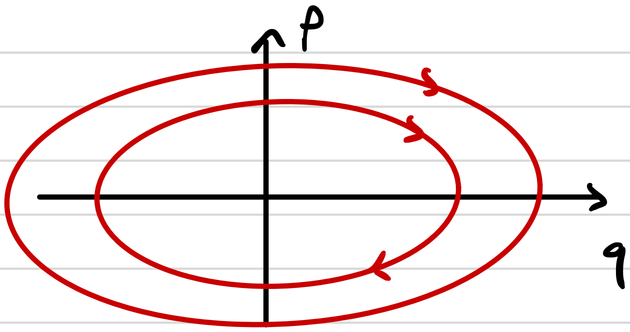
Natural choice of coordinates that makes solving a given problem easier?

$\hookrightarrow$ "Angle-Action" variable

Example : Harmonic Oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2$$

$$\left.\begin{aligned} \dot{p} &= -\frac{\partial H}{\partial q} = -m\omega^2 q \\ \dot{q} &= p/m \end{aligned}\right\} \quad \begin{aligned} q &= A\cos(\omega(t-t_0)) \\ p &= -m\omega A \sin(\omega(t-t_0)) \end{aligned}$$



E = constant

trajectories = ellipses

$(q,p) \longmapsto (\vartheta, I)$ (mixes q and p!)

$$q = \sqrt{\frac{2I}{m\omega}} \sin\vartheta \qquad p = \sqrt{2Im\omega}\cos\vartheta$$

Check that this is a canonical transformation:

$$\{q,p\}_{(\vartheta,I)} = \frac{\partial q}{\partial \vartheta}\frac{\partial p}{\partial I} - \frac{\partial q}{\partial I}\frac{\partial p}{\partial \vartheta}$$

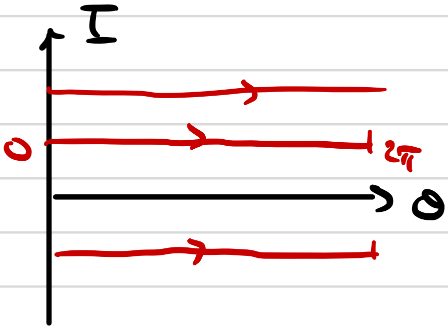
$$= \sqrt{\frac{2I}{m\omega}} \cos^2\vartheta \, \frac{\sqrt{2\omega m}}{2\sqrt{I}} + \frac{1}{2}\sqrt{\frac{1}{m\omega I}} \sin^2\vartheta \, \sqrt{2Im\omega}$$

$$= \cos^2\vartheta + \sin^2\vartheta = 1$$

$$H = \frac{1}{2m}(2m\omega I)\sin^2\vartheta + \frac{1}{2}m\omega^2 \frac{2I}{m\omega}\cos^2\vartheta$$

$$= \omega I$$

$$\Rightarrow \quad \dot{\vartheta} = \frac{\partial H}{\partial I} = \omega \qquad \dot{I} = -\frac{\partial H}{\partial \theta} = 0$$



⇒ we straightened out the flow lines!

Integrable systems

Can we do this generally?

N degrees of freedom $(q_i, p_i) \longmapsto (\theta_i, I_i)$ canonical such that $H(I_1, \dots, I_N)$ doesn't depend on ϑ_i.

If exists: Integrable systems $\qquad I_i$: conserved quantities

while $\quad \dot{\theta}_i = \frac{\partial H}{\partial I_i} = \omega_i \leftarrow$ depends on $\{I_j\}$

$$\Rightarrow \quad \vartheta_i = \omega_i t$$

For bounded motion, take $0 \le \vartheta_i < 2\pi$.

<u>Theorem</u>: (Liouville) If we can find N mutually Poisson commuting constants of motion: $\{I_i, H\} = 0$

$$\{I_i, I_j\} = 0$$

then the system is integrable ($\exists$ angle-action variables)

$$\frac{dI_i}{dt} = \{I_i, H\} + \frac{\partial I_i}{\partial t} = 0$$

Trajectory: $S^1 \times \dots \times S^1$. Invariant Tori.

Generalize to Quantum systems? $[\hat{H}, \hat{Q}_i] = 0$

$$[\hat{Q}_i, \hat{Q}_j] = 0$$

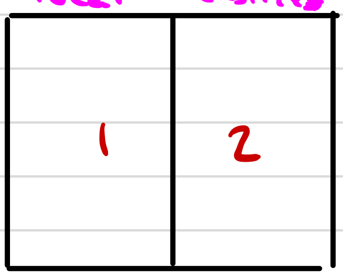
eigenstate

Problem: $\hat{Q}_i = |i\rangle\langle i|$ works for any quantum system!!

Locality: For many-body systems with local interactions, we also require that the conserved quantities be local
(sum of local densities)

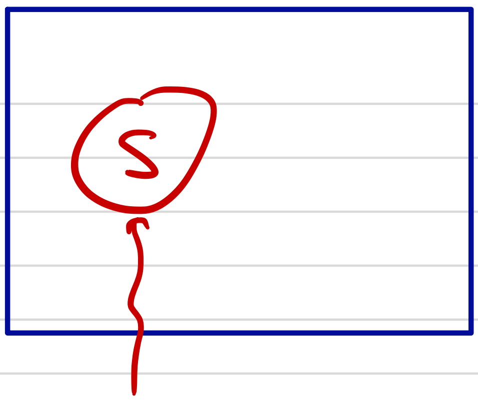
$E = \sum_i E_{i,i+1}$, $S^z = \sum_i s_i^z$ etc ...

Additivity: $E = E_1 + E_2 + \cancel{E_{12}}$
neglect



Generalized Thermalization

Under their own dynamics, such systems locally reach a generalized equilibrium state



subsystem

P_n = proba to be in state n.

$$Q_i = \sum_n Q_i^n P_n$$

conserved quantity (Energy, but many others for integrable systems)

Maximize entropy : $S = -\sum_n P_n \log P_n$

with constraints: $\sum_n P_n = 1$

$$\sum_n P_n Q_i^n = \langle Q_i \rangle$$

Enforces conservation laws

$$\delta\left(-\sum_n P_n \log P_n - \alpha\left(\sum_n P_n - 1\right) - \sum_j \beta_j\left(\sum_n Q_n^j P_n - \langle Q^j \rangle\right)\right) = 0$$

$$\Rightarrow -\sum_n \delta P_n \log P_n + \delta P_n + \alpha\, \delta P_n + \sum_j \beta_j Q_n^j \delta P_n$$

$$- \delta\alpha \underbrace{\left(\sum_n P_n - 1\right)}_{0} - \delta\beta_j \underbrace{\left(\sum_n Q_n^j P_n - \langle Q^j \rangle\right)}_{0} = 0$$

constraints

$$\sum_n \left(\left(\log P_n + 1 + \alpha + \sum_j \beta_j Q_n^j\right) \delta P_n = 0\right.$$

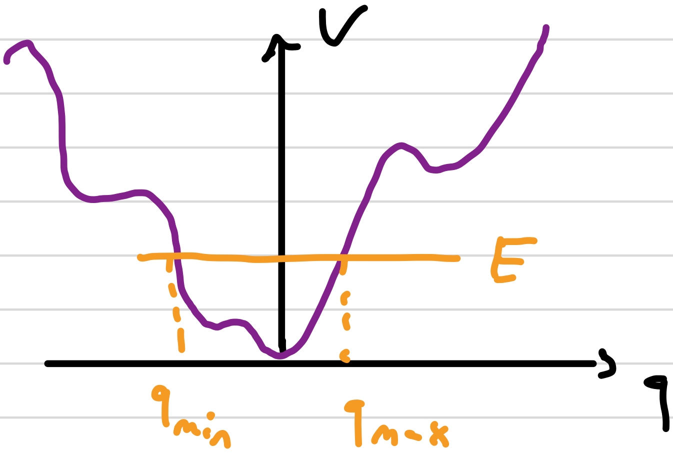
$$\Leftrightarrow \quad \boxed{P_n = \frac{1}{Z} e^{-\sum_j \beta_j Q_n^j}}$$

Generalized Gibbs Ensemble

with $Z = \sum_n e^{-\sum_j \beta_j Q_n^j}$

Action-angle variables for 1d systems

$H = \frac{p^2}{2m} + V(q) = E$: integrable since E is conserved



$$\dot\theta = \frac{\partial H}{\partial I} = \frac{\partial E}{\partial I} = \omega \ ?$$

$$I = f(E)$$

we have $p = \sqrt{2m(E-V)}$
$= m\dot q$

$$dt = \sqrt{\frac{m}{2}}\,\frac{dq}{\sqrt{E-V(q)}} \implies T = \frac{2\pi}{\omega} = \sqrt{\frac{m}{2}}\; 2\int_{q_{min}}^{q_{max}} \frac{dq}{\sqrt{E-V}}$$

$$2\int_{q_{min}}^{q_{max}} = \oint$$

$$= \sqrt{\frac{m}{2}} \oint 2\,\frac{d}{dE}\left(\sqrt{E-V}\right) dq$$

$$= \frac{d}{dE}\left(\oint \sqrt{2m(E-V)}\, dq\right) = \frac{d}{dE}\oint p\, dq$$

not entirely trivial
change in area due to change in $q_{min}, q_{max} = \mathcal{O}(dE^2)$

Let $\boxed{I = \frac{1}{2\pi}\oint p\, dq}$: $\frac{dE}{dI} = \omega$

and $t = \frac{d}{dE}\int p\, dq$: $\theta = \omega t = \frac{dE}{dI}\frac{d}{dI}\int p\, dq = \frac{d}{dI}\int p\, dq$

very brief

Ⅰ Hamilton-Jacobi and relation to quantum mechanics

function $\qquad$ functional

Hamilton-Jacobi equation:

$$S(\vec{q}_i, \vec{q}_f, t) = S[q] = \int_0^t L\, dt$$

solution of Euler Lagrange

$$\delta S = \int_0^t dt \underbrace{\left(\frac{\partial L}{\partial \vec{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{q}}}\right)\right)}_{0 \text{ for } q = \text{ solution of e.o.m.}} \cdot \delta \vec{q}(t) + \left[\frac{\partial L}{\partial \dot{\vec{q}}} \delta \vec{q}\right]_0^t$$

$$\frac{\partial S}{\partial \vec{q}_f} = \frac{\partial L}{\partial \dot{\vec{q}}}\bigg|_t = \vec{p}_f$$

Let's fix $\vec{q}_i$ = initial $\vec{q}$

$$\frac{dS}{dt} = \partial_t S + \frac{\partial S}{\partial \vec{q}_f} \cdot \dot{\vec{q}}_f = \partial_t S + \vec{p}_f \cdot \dot{\vec{q}}_f$$

and $\quad \dfrac{dS}{dt} = L\big|_t = L(\vec{q}_f, \dot{\vec{q}}_f, t)$

$$\Rightarrow \partial_t S = L_f - \vec{p}_f \cdot \dot{\vec{q}}_f = -H_f = -H(\vec{q}_f, \vec{p}_f, t)$$

we have: $\boxed{\dfrac{\partial S}{\partial t} + H\left(\vec{q}, \dfrac{\partial S}{\partial \vec{q}}, t\right) = 0}$

drop "final": $\vec{q}_f \to \vec{q}$

Let's say we solve this eq: $S(\vec{q},t)$

then: $\dot{\vec{q}} = \left.\frac{\partial H}{\partial \vec{p}}\right|_{\vec{p}=\frac{\partial S}{\partial \vec{q}}}$ $\to$ N 1^{st} order eqs. for $\vec{q}(t)$

Check: $\dot{\vec{p}} = \frac{d}{dt}\left(\frac{\partial S}{\partial \vec{q}}\right) = \frac{\partial^2 S}{\partial t \partial \vec{q}} + \frac{\partial^2 S}{\partial \vec{q}^2}\cdot\dot{\vec{q}}$

But $\underset{\uparrow}{\frac{\partial S}{\partial t \partial \vec{q}}} = -\frac{\partial H}{\partial \vec{q}} - \underbrace{\frac{\partial H}{\partial \vec{p}}}_{\dot{\vec{q}}}\cdot\frac{\partial^2 S}{\partial \vec{q}^2}$: $\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{q}}$

Hamilton Jacobi

- Lagrange : N 2^{nd} order equations
- Hamilton : 2N 1^{st} order equations
- Hamilton-Jacobi: $S(\vec{q},t)$ incorporates N initial conditions $\vec{q}_i$

and then N 1^{st} order equations

if $\partial_t H = 0$: Solution $S = S_0(\vec{q}) - Et$

$$\boxed{H\left(\vec{q}, \frac{\partial S}{\partial \vec{q}}\right) = E}$$

We won't spend more time on this approach here.

Relations to quantum

Canonical quantization: $\boxed{\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]}$

$$\{q_i, p_j\} = \delta_{ij} \Rightarrow [\hat{q}_i, \hat{p}_j] = i\hbar\,\delta_{ij}$$

$$\dot{f} = \{f, H\} \Rightarrow i\hbar\,\dot{\hat{f}} = [\hat{f}, \hat{H}]$$

Schrödinger eq: $i\hbar\,\partial_t \psi = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\psi$

with $p = -i\hbar\frac{\partial}{\partial q}$. Plug: $\psi(q,t) = \rho(q,t)\, e^{i\theta(q,t)/\hbar}$

as $\hbar \to 0$: $\frac{\partial \theta}{\partial t} + \frac{1}{2m}\left(\frac{\partial \theta}{\partial q}\right)^2 + V(q) = 0$

Hamilton Jacobi!

In classical limit: Phase of wave function = classical action of classical trajectory!

In general: $\psi(q_f, t) \sim \int_{q_i}^{q_f} \mathcal{D}q(t)\; e^{iS[q(t)]/\hbar}$

path integral

classical path dominates as $\hbar \to 0$