Classical Field Theory

(Very brief)

Coupled springs: (in 1d chain) $L = \sum_i \frac{1}{2} m \dot{\delta}_i^2 - \sum_i \frac{1}{2} K(\delta_{i+1} - \delta_i)^2$

δ_i = deviation from equilibrium position

Continuum limit: $\sum_i \longrightarrow \int \frac{dx}{a}$ $\qquad a$ = lattice spacing

$$\delta_i(t) = \phi(x_i = i\,a, t)$$

$$\delta_{i+1} - \delta_i = \phi((i+1)\,a) - \phi(i\,a)$$
$$= \phi(x_i + a) - \phi(x_i) \simeq a\, \partial_x \phi \big|_{x_i}$$

$$\Rightarrow \quad L = \int \frac{dx}{a} \left[\frac{1}{2} m (\partial_t \phi)^2 - \frac{1}{2} K a^2 (\partial_x \phi)^2 \right]$$
$$= \int \frac{dx}{2a} m \left[(\partial_t \phi)^2 - \frac{Ka^2}{m} (\partial_x \phi)^2 \right]$$

v^2 : v = wave velocity

set $a=1$

Now action: $S[\beta] = \int dt\, dx\, \mathcal{L}(\beta, \partial_t \beta, \partial_x \beta)$

$\beta = \phi$ for us

Lagrangian density

$$\delta S = \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial \beta} \delta \beta + \frac{\partial \mathcal{L}}{\partial(\partial_t \beta)} \delta \partial_t \beta + \frac{\partial \mathcal{L}}{\partial(\partial_x \beta)} \delta \partial_x \beta \right)$$

$$= \int dt\, dx \left(\frac{\partial \mathcal{L}}{\partial \beta} - \partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \beta)} \right) - \partial_x \left(\frac{\partial \mathcal{L}}{\partial(\partial_x \beta)} \right) \right) \delta \beta$$

$\Rightarrow$ Euler - Lagrange: $\boxed{\frac{\partial \mathcal{L}}{\partial \beta} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \beta)} \right)}$

$\mu = x, t$: summed over

Here: $\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = m\, \partial_t \phi$

$$\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} = -m v^2 \partial_x \phi$$

$\Rightarrow$ $\boxed{\partial_t^2 \phi - v^2 \partial_x^2 \phi = 0}$ wave equation

Hamiltonian: $\pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = m\, \partial_t \phi$: conjugate momentum

$$H = \int dx \left(\pi(x,t)\, \partial_t \phi \right) - L = \int dx\, \mathcal{H}$$

with:

$$\mathcal{H} = \pi\, \partial_t \phi - \mathcal{L}$$

$$= \frac{\pi^2}{m} - \frac{m}{2}\left[(\underbrace{\partial_t \phi}_{\pi^2/m})^2 - v^2 (\partial_x \phi)^2 \right]$$

$$\boxed{\mathcal{H} = \frac{\pi^2}{2m} + \frac{K}{2}(\partial_x \phi)^2}\,, \quad H = \int dx\, \mathcal{H}$$

$$\{\phi(x), \pi(y)\} = \delta(x-y)$$

where $\{f, g\} = \int dx \left(\dfrac{\delta f}{\delta \phi(x)} \dfrac{\delta g}{\delta \pi(x)} - \dfrac{\delta f}{\delta \pi(x)} \dfrac{\delta g}{\delta \phi(x)} \right)$

$$\left(\frac{\delta \phi(x)}{\delta \phi(y)} = \delta(x-y) \quad \leftrightarrow \quad \frac{\partial q_i}{\partial q_j} = \delta_{ij} \right)$$