Microhydrodynamics of soft particles

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BROWN

Fluids at Brown

Complex Fluids

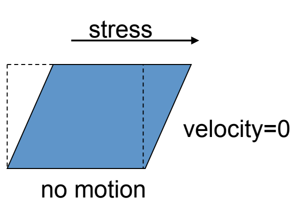
- Detergents
- Paints
- Plastics
- Food
- Cosmetics
- All living systems

A classic example:

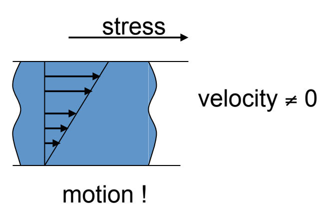
"solid" at short times
"liquid" at long times

Complex fluids: neither solids, nor fluids

solid under static shear



fluid under static shear



$$\tau_{xy} = G\epsilon \qquad \tau_{xy} = \mu\dot{\gamma}$$

A solid resists a shear stress by a static deformation (it stores elastic energy); a fluid can not (it dissipates energy via internal friction)

For simple materials the coefficient of proportionality between stress and strain/rate of strain is elasticity/viscosity; they are "constants"

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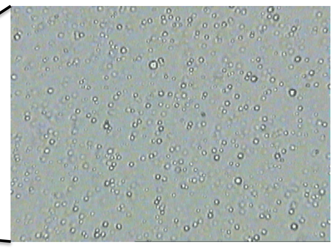
Complex fluids are structured

micronsized particles
embedded in fluid/water
flow
particles may deform in flow

Microhydrodynamics of soft particles

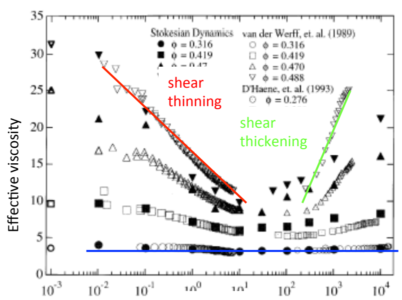
Zoom x 1000



milk

drop size ~ microns

Even suspensions of solid particles (e.g. paint) show complex rheology!



J.F. Brady (2001)

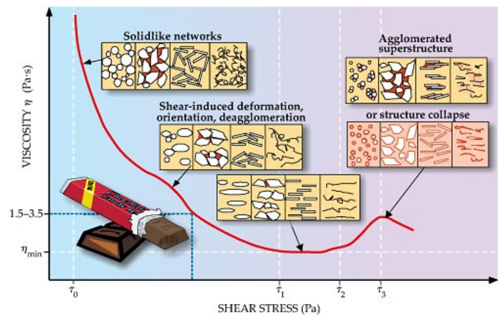
"solid" at short times
"liquid" at long times
(time scales: flow vs microstructure relaxation

Pe<1 Brownian motion important
Pe>1 Brownian motion disappears, "strings" parallel to the flow direction : shear thinning
Pe>>1 jamming (particles stuck together by strong lubrication forces) : shear thickening

My favorite complex fluid:

What makes for a smooth, creamy chocolate?

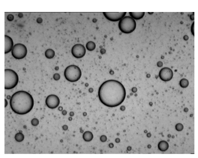
Physics Today, June 2006



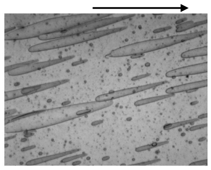
Chocolate: Cocoa butter, cocoa powder, sugar, milk powder, emulsifier,…

Soft matter/Complex fluids: suspensions of soft particles

emulsions, polymer blends



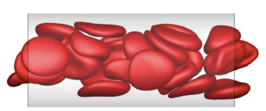
shear flow



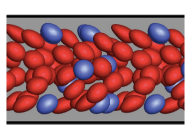
Tucker et al. Annu Rev Fluid Mech (2002)

blood

simulation of RBCs in capillary flow

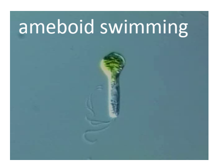


Zhao and Freund, J Comp Phys (2010)
Freund Annu Rev Fluid Mech (2014)

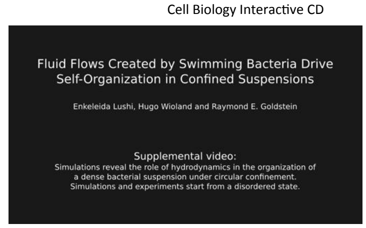


Kumar and Graham PRE (2011)
(segregation by capsule deformability)

cells



Cell Biology Interactive CD



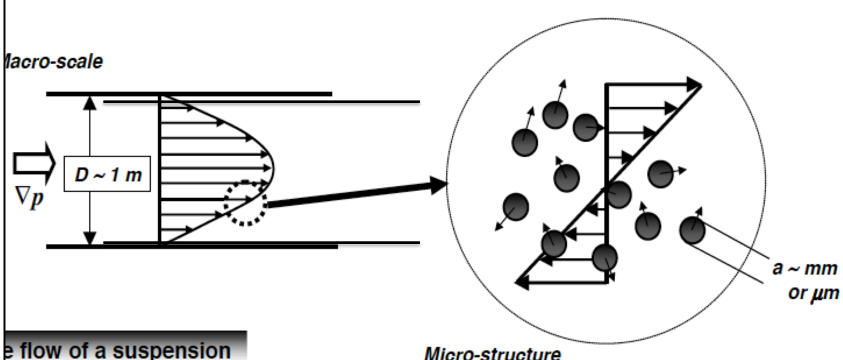
Lushi et al PNAS (2014)

deformability ↔ effective macroscopic properties (viscosity, conductivity), collective behavior (patterns, mixing…)

Complex fluid: continuum description

Effective properties determined by the microstructure

Deformable microstructure: complex rheology

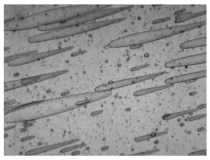


Picture from notes by: Ganesh Subramanian

fluid (continuous) phase: conservation of mass, momentum
disperse(particle) phase: microhydrodynamics, Brownian motion

particle length scale<< flow scale => effective fluid properties by averaging over the small scales

shear flow →



Tucker et al. (2002)

stress $\mathbf{T} = -p\mathbf{I} + \tau$

$\tau(\mathbf{E})$ traceless => 5 independent components: 3 shear, 2 normal stresses

Increasing number of degrees of freedom:=> complex, non-Newtonian rheology

- position
- position and orientation
- deformable microstructure –polymers, "soft" particles (importance of interfaces), active particles, EDL

Examples of NN phenomena:

1) Non-linear flows

$$Q = \left(\frac{\pi R^4}{8L\mu}\right)\Delta P$$

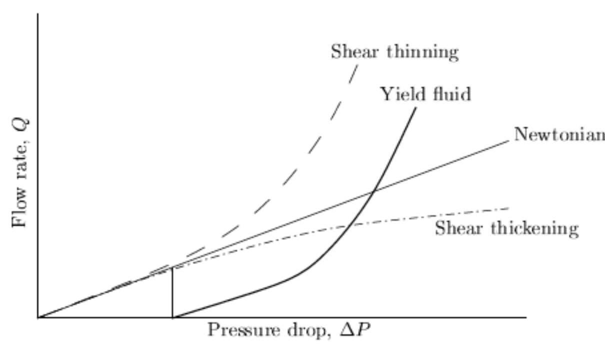


Figure 1: Flow rate as a function of pressure drop for flow down a pipe.

Hinch, lecture notes Woods Hole 2003

Rheology – important for material processing

Fåhraeus and Lindqvist (1931)

apparent viscosity

$$\mu = \left(\frac{\pi R^4}{8L}\right)\frac{\Delta P}{Q}$$

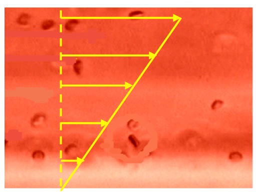
THE VISCOSITY OF THE BLOOD IN NARROW CAPILLARY TUBES

ROBIN FÅHRÆUS AND TORSTEN LINDQVIST

From the Pathological Institute, Uppsala, Sweden

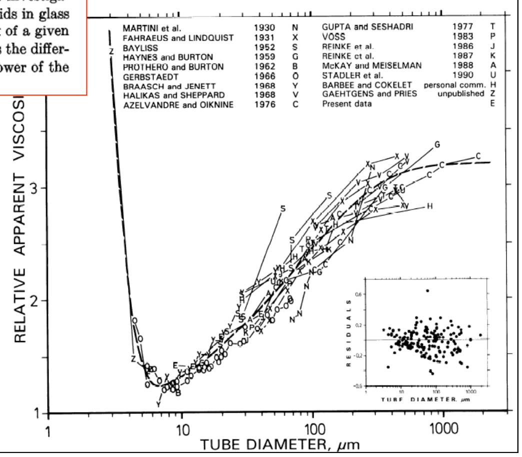
Received for publication December 6, 1930

Nearly one hundred years have passed since the French physician Poiseuille (1) took up for consideration the important problem of the resistance of the bloodstream in the narrow parts of the vascular system. As experimental difficulties arose with blood, his fundamental investigations were confined to experiments with water and different fluids in glass capillaries. He found, as is well known, that the time of efflux of a given volume of fluid is directly as the length of the tube, inversely as the difference of pressure at the two ends and inversely as the fourth power of the diameter.



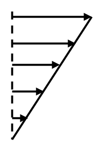
RBC depleted layer near the wall

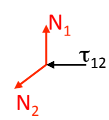
A. Viallat

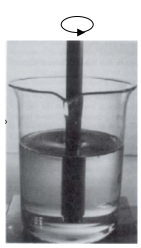


2) Normal stresses effects

can be viewed as due to tension in the streamlines





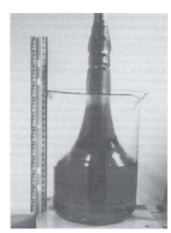


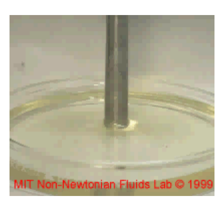
Zero normal stresses
→ Newtonian fluid

$$N_1 = N_2 = 0$$

Nonzero normal stresses
→ Non-Newtonian fluid

$$N_1 \neq 0 \quad N_2 \neq 0$$





rod-climbing

Hoop stress

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Normal stress effects on distribution of particles:
particle aggregation in sheared polymer solutions

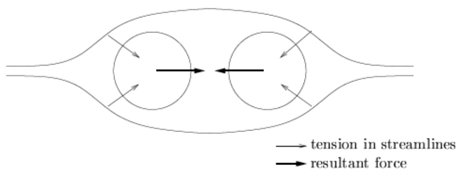
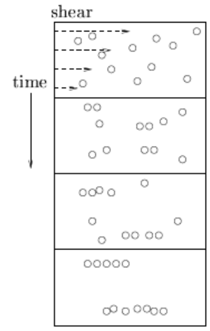


Figure 9: Balance of forces for two particles in a simple shear.



cross- stream migration

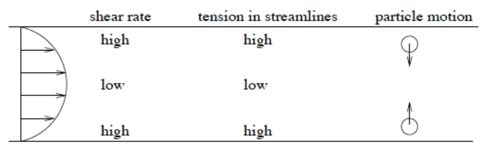


Figure 11: Migration of particles to centerline in a non-Newtonian pipe flow.

Hinch, lecture notes Woods Hole 2003

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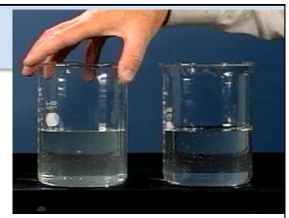
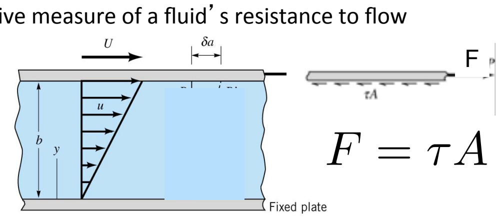
Rheology: viscosity



quantitative measure of a fluid's resistance to flow



$$F = \tau A$$

τ : shear stress

$$\tau = \mu \dot{\gamma}$$

shear rate (rate of strain)

$$\dot{\gamma} = \frac{du}{dy}$$

For a Couette flow

$$\frac{du}{dy} = \frac{U}{b}$$

μ : (dynamic) viscosity

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Shear flow driven by rotation of the cylinder wall (flow is visualized by a dye line)

Which fluid is more viscous?

A. The red fluid is more viscous than the yellow one
B. The yellow fluid is more viscous than the red one
C. Both fluids have same viscosity
D. Not enough information

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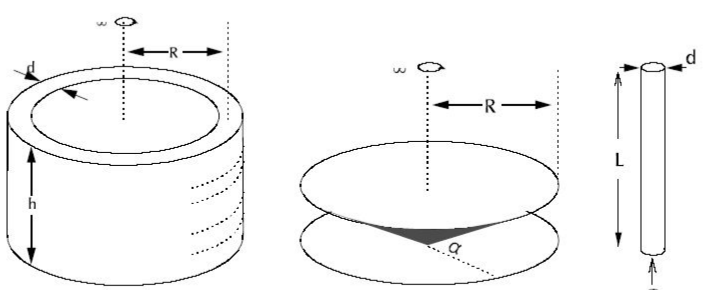
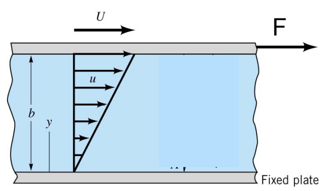
Viscometers

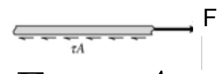


$$\mu = \frac{M\, d}{2\, \pi\, \omega\, R^3\, h} \qquad \mu = \frac{3\, \alpha\, M}{2\, \pi\, \omega\, R^2} \qquad \mu = \frac{\Delta P\, \pi\, d^4}{128\, L\, Q}$$

Figuur 8.3: Drie verschillende viscosimeters

What is the relation between the force and velocity of the top plate?





$$F = \tau A$$

$$\tau = \mu \frac{du}{dy} = \mu \frac{U}{b}$$

$$F = \frac{\mu U A}{b} \qquad U = \frac{F b}{\mu A}$$

The higher the fluid viscosity, the stronger the opposing friction force

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The problem: Rheology of dispersions

Effective viscosity for a dilute dispersion with particle volume fraction ϕ

$$\mu_{\text{eff}} = \tau / \dot{\gamma} = \mu_0 \left(1 + \phi \eta^d\right)$$

• suspension of rigid spheres — Newtonian — $\eta^{\mathrm{d}} = \frac{5}{2}$

Einstein (1905)

drop deformation?
surfactant?
Interfacial viscosity?

Non-Newtonian ?

• emulsion of spherical "clean" drops — $\eta^{\mathrm{d}} = \frac{1 + \frac{5}{2}\lambda}{1 + \lambda}$

G.I.Taylor (1934) — Newtonian

λ viscosity ratio

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Lecture 1: intro, complex fluids properties, complex fluids as continuum, effective viscosity
Lecture 2: equations,, stress tensor, nondimensionalization
Lecture 3: Stokes flow: basic properties, fundamental solutions, stresslet
Lecture 4: Rheology of dispersions of soft particles: examples

Microhydrodynamics of soft particles

Petia M. Vlahovska, Brown university

Lecture 2: Fundamentals of Viscous Fluid Dynamics

1 The fluid as continuum. Conservation of mass and momentum - Reynolds Transport Theorem

$$\frac{D}{Dt}\int_V \rho\psi dV = \frac{\partial}{\partial t}\int_V \rho\psi dV + \int_S \rho\psi \mathbf{u}\cdot\mathbf{n} dS = \int_V \left(\frac{\partial}{\partial t}(\rho\psi) + \nabla\cdot(\rho\psi\mathbf{u})\right) dV \tag{1}$$

For mass $\psi = 1$, for linear momentum $\psi = \mathbf{u}$, for angular momentum $\psi = \mathbf{r}\times\mathbf{u}$.

a). continuity equation $\frac{D}{Dt}\int_V \rho dV = 0$

b). linear momentum conservation: $\frac{D}{Dt}\int_V \rho\mathbf{u} dV = forces$; body and surface forces; stress tensor (need constitutive law relating stress to velocity)

2 Stress: constitutive laws

a) conservation of angular angular momentum: show that if there are no couples (torques) $\Longrightarrow$ stress tensor is symmetric. Example of a fluid with non-symmetric stress tensor: ferrofluid

b) Newtonian fluid: linear relation between stress and velocity field (constitutive law). Stress tensor is symmetric.

$$\mathbf{T} = (-p + (\kappa + 2/3\mu)\nabla\cdot\mathbf{u})\,\mathbf{I} + 2\mu\mathbf{E}$$

κ is bulk viscosity, μ is shear viscosity, $\mathbf{E}$: rate-of-strain tensor, It is symmetric and traceless, $\mathbf{E} = [\nabla\mathbf{u}]^{sym}$, $E_{ij} = 1/2\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_i}{\partial x_i}\delta_{ij}\right)$. Note that $Tr(\mathbf{E}) = \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial v_i}{\partial x_i} - \frac{2}{3}\frac{\partial v_i}{\partial x_i}\delta_{ii}\right) = 0$ $(\delta_{ii} = 3)$

$\bar{p} = -1/3Tr(\mathbf{T}) = p - (\kappa + 2/3\mu)\nabla\cdot\mathbf{u}$; $\bar{p}$ is the mechanical pressure

c) Equation of motion for an incompressible, Newtonian fluid: *Navier-Stokes equations*

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \quad \nabla\cdot\mathbf{u} = 0 \tag{2}$$

$P = p - \rho g z$ - dynamic pressure, usually the $\rho\mathbf{g}$ term absorbed in the pressure

$$\rho\left(\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla P + \mu\nabla^2\mathbf{u} \quad \nabla\cdot\mathbf{u} = 0 \tag{3}$$

3 Nondimensionalization

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \tilde{p} = \frac{p}{p_c}, \quad \tilde{t} = \frac{t}{t_c}$$

$$Re\left(\frac{1}{St}\frac{\partial\tilde{\mathbf{u}}}{\partial\tilde{t}} + \tilde{\mathbf{u}}\cdot\tilde{\nabla}\tilde{\mathbf{u}}\right) = -\frac{p_c}{\mu U/L}\tilde{\nabla}\tilde{p} + \tilde{\nabla}^2\tilde{\mathbf{u}} \tag{4}$$

$$\text{Reynolds number } Re = \frac{\rho U L}{\mu} = \frac{inertia}{viscosity} = \frac{t_v}{L/U}$$

$t_v = \frac{L^2}{\nu}$ is the viscous (momentum diffusion) time scale ($\nu = \mu/\rho$ is the kinematic viscosity)
L/U is the inertia (flow) time scale

$$\text{Strouhal number } St = \frac{t_c}{L/U} \quad \text{for example } t_d = \omega^{-1}(\text{period of oscillations in oscillatory shear})$$

limits:

balance viscous forces and pressure:

$$Re \ll 1 \text{ and } Re/St \ll 1 \text{ Stokes flow } p_c = \mu U/L \rightarrow \text{Stokes equation}$$

however

$$Re \ll 1 \text{ but } Re/St \sim 1 \text{ unsteady viscous flow } p_c = \mu U/L \rightarrow \text{unsteady Stokes equation}$$

balance inertia and pressure:

$$\text{if } Re \gg 1 \text{ and steady flow } \partial/\partial t = 0 \rightarrow \text{ inviscid flow } p_c = \rho U^2 \rightarrow \text{Bernouli equation}$$

other important dimensionless parameter is the Peclet number

$$Pe = \frac{UL}{D_p} = \frac{L^2/D_p}{L/U} = \frac{\text{particle diffusion time scale}}{\text{flow(convection) time scale}}$$

Complex fluids are made of small, colloidal particles. Choosing the characteristic length scale L to be the particle size (e.g. for a bacterium $L = 1\mu m$) shows that the flow as "seen" by the particle is dominated by viscosity (for water $\rho = 1000kg/m^3$, $\mu = 10^{-3}Pa.s$) if $U \ll 1m/s$. If U is the bacterium swimming speed $U \sim 1\mu m/s$ then $Re \sim 10^{-6} \ll 1$. Thus we can immediately estimate the order of magnitude of the drag force on a colloidal particle -it is the magnitude of the viscous stresses p_c multiplied by the particle area L^2, $F_D \sim p_c L^2$. $F_d \sim \mu U L$. The exact solution for the viscous flow past a sphere with radius L gives us the numerical pre factor $F_D = 6\pi\mu U L$

Peclet number measures the importance of Brownian motion relative to particle advection by the flow. For sheared suspensions (see movie from lecture 1), $L/U = 1/shear\,rate = \dot{\gamma}^{-1}$. If $Pe \leq 1$ then Brownian motion is important (the particle diffuses faster than the fluid moves, and as a result the particle will not follow the flow streamlines!!!!)

[SHOW MOVIES: illustrating regimes of low/high Re, low/high Pe]

Lecture 3. Stokes flow and solutions; Boundary integral formulation, multipole explansion

References:

C. Pozrikidis " Boundary integral and singularity methods for linearized viscous flow", 1992
Kim and Karrila "Microhydrodynamics"

4 Stokes Equations: properties, Fundamental solutions

Linearity and reversibility. examples: cross-stream migration

Stokeslet: flow due to a point force with strength $\mathbf{F}$ ($\mathbf{F}$ is force ON the fluid; it is minus the force exerted by the fluid on the body)

$$\nabla \cdot \mathbf{T} = \mu \nabla^2 \mathbf{u} - \nabla p = -\mathbf{F}\delta(\mathbf{x}) \tag{5}$$

i.e., for $\mathbf{x} \neq 0\,,\quad \nabla \cdot \mathbf{T} = 0$ and for any volume V that encloses the point $\mathbf{x} = 0$ $\int \nabla \cdot \mathbf{T} dV = -\mathbf{F}$.

Solution:

$$u_i = \frac{1}{8\pi\mu} G_{ij} F_j\,, \quad p = \frac{1}{8\pi} P_i F_i \tag{6a}$$

$$G_{ij}(\mathbf{x}) = \frac{\delta_{ij}}{r} + \frac{x_i x_j}{r^3} \qquad P_i(\mathbf{x}) = 2\frac{x_i}{r^3} \tag{6b}$$

exercise: Substitute Eq. (6) into Eq. (5) to show that Eq. (6) is a solution. Show that the Green's dyadic G_{ij} also satisfies the continuity equation, $\nabla \cdot \mathbf{u} = 0$.

The stress field is

$$T_{ik} = \frac{1}{8\pi} \Sigma_{ijk} F_j \quad \text{where} \quad \Sigma_{ijk} = -P_j \delta_{ik} + \frac{\partial G_{ij}}{\partial x_k} + \frac{\partial G_{kj}}{\partial x_i} = -6 \frac{x_i x_j x_k}{r^5} \tag{7}$$

(Verify that $\int \nabla \cdot \mathbf{T} dV = -\mathbf{F}$.)

Boundary Integral Equation:

Let ($\mathbf{u}$, $\mathbf{T}$) is a solution (velocity, stress tensor) of the Stokes equations and ($\mathbf{u}_s, \mathbf{T}_s$) is the Stokeslet solution, for a point force, $\nabla_\xi \cdot \mathbf{T}_s = -\mathbf{g}\delta(\boldsymbol{\xi} - \mathbf{x})$ ($\mathbf{g}$ is a constant vector). $\mathbf{u}$ decays at infinity, so if there is a flow applied at infinity $\mathbf{u}$ stands for the disturbance velocity field, $\mathbf{u} - \mathbf{u}^\infty$. Start with the Lorentz identity (reciprocal theorem)

$$\int_S (-\mathbf{u} \cdot \mathbf{T}_s + \mathbf{u}_s \cdot \mathbf{T}) \cdot \mathbf{n} dS = \int_V (-\mathbf{u} \cdot \nabla \cdot \mathbf{T}_s + \mathbf{u}_s \cdot \nabla \cdot \mathbf{T})\, dV \tag{8}$$

Steps:

- $$\int_V (-\mathbf{u} \cdot \nabla \cdot \mathbf{T}_s)\, dV = \mathbf{u}(\mathbf{x}) \cdot \mathbf{g}$$

- $$\nabla \cdot \mathbf{T} = 0$$

- for a solid particle the fluid velocity at the surface is zero (no slip)

$$\int_S (-\mathbf{u} \cdot \mathbf{T}_s) \cdot \mathbf{n} dS = 0$$

- since $\mathbf{T} \cdot \mathbf{n} = \mathbf{f}$, $\mathbf{u}_s = c\mathbf{G} \cdot \mathbf{g}$ ($c = 1/8\pi\mu$),

$$\int_S (\mathbf{u}_s \cdot \mathbf{T}) \cdot \mathbf{n} dS = \int_S (\mathbf{G} \cdot \mathbf{g}) \cdot \mathbf{f} dS$$

- so we get

$$\mathbf{u}(\mathbf{x}) \cdot \mathbf{g} = c \int_S (\mathbf{G} \cdot \mathbf{g}) \cdot \mathbf{f} dS$$

- since $\mathbf{g}$ is arbitrary constant vector it cancels on both sides and we get

$$\mathbf{u}(\mathbf{x}) = c \int_S (\mathbf{G}) \cdot \mathbf{f} dS$$

In general, (when $\mathbf{u} \neq 0$ on the particle surface) (note the normal $\mathbf{n}$ points out of the particle, into the fluid region)

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}^\infty = -\frac{1}{8\pi\mu} \int_S \mathbf{G}(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) - \frac{1}{8\pi} \int_S \mathbf{u}(\boldsymbol{\xi}) \cdot \boldsymbol{\Sigma}(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{n} dS(\boldsymbol{\xi}) \tag{9}$$

first term on the RHS is the single-layer potential, and the second - the double-layer potential. Note $G_{ij}(\mathbf{x} - \boldsymbol{\xi}) = G_{ij}(\boldsymbol{\xi} - \mathbf{x})$ but $\Sigma_{ijk}(\mathbf{x} - \boldsymbol{\xi}) = -\Sigma_{ijk}(\boldsymbol{\xi} - \mathbf{x})$. $\mathbf{f} = \mathbf{T} \cdot \mathbf{n}$. and for a solid particle

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}^\infty = -\frac{1}{8\pi\mu} \int_S \mathbf{G}(\mathbf{x} - \boldsymbol{\xi}) \cdot \mathbf{f}(\boldsymbol{\xi}) dS(\boldsymbol{\xi}) \tag{10}$$

The multipole expansion:

Far away from the particle $|\mathbf{x}| \gg |\boldsymbol{\xi}|$, take Taylor series in $\boldsymbol{\xi}$ about $\boldsymbol{\xi} = 0$

$$G_{ij}(\mathbf{x} - \boldsymbol{\xi}) = G_{ij}(\mathbf{x}) - \boldsymbol{\xi} \cdot \nabla_\xi G_{ij}(\mathbf{x} - \boldsymbol{\xi})|_{\boldsymbol{\xi}=0} + ... \tag{11}$$

then

$$u_i(\mathbf{x}) - u_i^\infty = -\frac{F_j}{8\pi\mu} G_{ij}(\mathbf{x}) + \frac{D_{jk}}{8\pi\mu} \frac{\partial G_{ij}}{\partial x_k} + ... \tag{12}$$

The antisymmetric part of D_{jk} can be identified with the hydrodynamic torque on the particle, $\mathbf{L} = \int \mathbf{x} \times \mathbf{f} dS$

$$D_{jk}^{as} = \frac{1}{2} \int [f_j \xi_k - f_k \xi_j] \, dS = -\frac{1}{2} \varepsilon_{jkn} L_n \quad \text{or} \quad L_n = -\varepsilon_{njk} D_{jk}^{as} \tag{13}$$

The symmetric part is the stresslet - point force dipole.

exercise: Show that

- $$\frac{\partial G_{ij}}{\partial x_k} = \frac{1}{r^3} (-\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ik} x_j) - \frac{3}{r^5} x_i x_j x_k$$

- the symmetric part is (stresslet flow)

$$\frac{\delta_{jk}x_i}{r^3} - \frac{3}{r^5}x_i x_j x_k$$

- the antisymmetric part is (rotlet flow)

$$\frac{1}{r^3}\left(-\delta_{ij}x_k + \delta_{ik}x_j\right)$$

- See that the rotlet flow is

$$D^{as}_{jk}\frac{1}{r^3}\left(-\delta_{ij}x_k + \delta_{ik}x_j\right) = L_j\left(-\varepsilon_{ijk}\frac{x_k}{r^3}\right)$$

Physical relevance:

1) flow due to a sedimenting sphere $\sim$ Stokeslet

2) micro swimmers (bacterial, algae) - force-free and torque-free. Flow field is approximated by stresslet .

For an experiment visualizing the flow around swimming microorganism see http://arxiv.org/pdf/1008.2681.pdf for further reading - see recent review by Eric Lauga "Bacterial Hydrodynamics" Annu. Rev. Fluid Mech. 2016. 48:10530

Lecture 4. part 1: Complex fluids -rheology

source of complex rheology: particle interactions, deformation

Effective stress

Normal stresses

If fluid motion is in the "1" direct and velocity gradient is in the "2" direction, then

$$N_1 = T_{11} - T_{22}\,, \quad N_2 = T_{22} - T_{33} = -\frac{1}{2}N_1 + \frac{1}{2}\left(T_{11} + T_{22} - 2T_{33}\right)$$

we used $T_{11} + T_{22} + T_{33} = 0$ (the stress tensor is traceless) to get the relation between N_1 and N_2

In shear flow $u_1 = \dot{\gamma} x_2$, $E_{ij} = \delta_{i1}\delta_{j2}$

$$T_{ij} = -p\delta_{ij} + 2\mu^{eff}\dot{\gamma}\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + N_1\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{2}{3}\left(N_2 + \frac{1}{2}N_1\right)\begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(so the shear stress $T_{12} = \mu^{eff}\dot{\gamma}$). For a Newtonian fluid in shear (Couette) flow $N_1 = 0$ and $N_2 = 0$

Rheology of a dilute suspension: [see e.g., Leal "Advanced Transport Phenomena" p.473 and Kim and Karrila "Microhydrodynamics", Example 2.1] The effective stress is the ensemble average of the stress distribution in all realizations of the suspension. For a homogeneous suspension, this ensemble average is equivalent to a volume average over a volume that is large enough to contain statistically significant number of particles

$$T^{eff}_{ij} = \frac{1}{V}\int T_{ij}dV = -\langle p\rangle\delta_{ij} + \frac{1}{V}\int_V (2\mu E_{ij}))dV + \frac{1}{V}\int_V (\tau_{ij} - 2\mu E_{ij}))dV \tag{14}$$

where $E_{ij} = 1/2\left(\partial u_i/\partial x_j + \partial u_j/\partial x_i\right)$. The last integral is identically zero for the fluid and reduces to an integral over the particles.

$$\int_{V_p} \tau_{ij}dV = \int_{S_p} f_i x_j dS \qquad \text{where} \quad f_i = \tau_{ik}n_k \tag{15}$$

The integral over the fluid region

$$\int_{V-\sum V_p} (2E_{ij})dV = \int_V (2E_{ij})dV - \sum_n \int_{V_p} (2E_{ij})dV \tag{16}$$

$$\int_{V_p} (2E_{ij})dV = \int_{S_p} (u_i n_j + u_j n_i)dS \tag{17}$$

Hence

$$\int_{V-\sum V_p} (2E_{ij})dV = 2\langle E_{ij}\rangle - \frac{1}{V}\sum_n \int_{S_p} (u_i n_j + u_j n_i)dS \tag{18}$$

We assume that particles contributions are additive - i.e., we neglect the hydrodynamic interactions (dilute suspension).

Putting it all together

$$T^{eff}_{ij} = -p^{eff}\delta_{ij} + 2\mu\langle E_{ij}\rangle + T^p_{ij} \tag{19}$$

where $\langle E_{ij} \rangle = E_{ij}^{\infty}$ (applied flow) and the particle stress is

$$T_{ij}^{p} = \frac{1}{V} \sum_{n} \int_{S_p} \left[f_i x_j - (u_i n_j + u_j n_i) \right] dS \tag{20}$$

decomposing into a symmetric traceless and an antisymmetric component:

$$T_{ij}^{p} = \frac{1}{V} \sum_{n} \left(\mathcal{S}_{ij} + \frac{1}{2} \epsilon_{ijm} L_m \right) \tag{21}$$

$\mathcal{S}$ is the stresslet and L is the torque on the particle.

For a **sphere in a shear flow** the velocity field is (Leal Eq. 8-44)

$$\mathbf{u} = \mathbf{u}^{\infty} - \mathbf{E} \cdot \mathbf{x} \frac{a^5}{r^5} - \frac{1}{2} \boldsymbol{\omega} \times \mathbf{x} \frac{a^3}{r^3} - \frac{5}{2} \mathbf{x} \left(\mathbf{x} \cdot \mathbf{E} \cdot \mathbf{x} \right) \left(\frac{a^3}{r^5} - \frac{a^5}{r^7} \right) \tag{22}$$

where $\mathbf{u}^{\infty} = \boldsymbol{\Gamma} \cdot \mathbf{x}$, $\Gamma_{ij} = \dot{\gamma} \delta_{i1} \delta_{j2} = E_{ij} + \epsilon_{ijk} \omega_k$

The stresslet coefficient is $\mathcal{S}_{ij} = 8\pi\mu a^3 \left(\frac{5}{6} E_{ij} \right)$. Thus the effective stress of a dilute suspension becomes

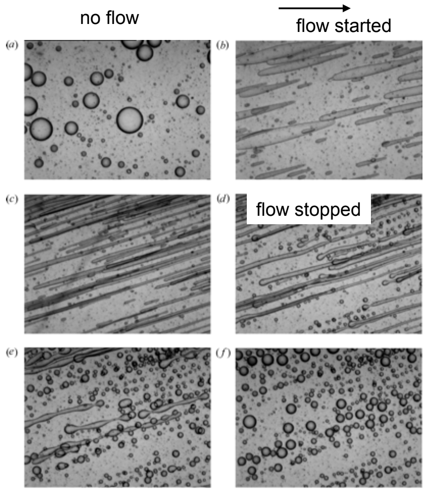
$$T_{ij} = -p^{eff} \delta_{ij} + 2\mu E_{ij} + \frac{N}{V} \mathcal{S}_{ij} = -p^{eff} \delta_{ij} + 2\mu E_{ij} \left(1 + \frac{5}{2} \phi \right) \qquad \phi = \frac{4\pi a^3 N}{V} \tag{23}$$

so the effective viscosity is $\mu^{eff} = \mu \left(1 + \frac{5}{2} \phi \right)$

part2: examples - powerpoint

Non-Newtonian rheology of dispersions of soft particles

Flow dependent microstructure



Iza&Bousmina (2000)

response to a perturbation depends on microstructure, which in turn depends on the perturbation

particles deform and align with the flow

(particle shape is not given a priori)

particle distribution becomes inhomogeneous

1

---

particle microhydrodynamics (microscale)	⟷	structure and flow of dispersions (macroscale)

Challenges:

- particle (drops, cells) deformability: interface important
- many-body (hydrodynamic) interactions

2

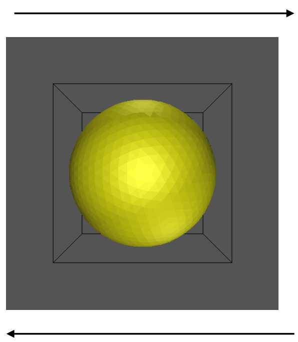
Soft particle: drop

Equilibrium:
drop shape is spherical

Laplace's equation

$$p_{in} - p_{out} = 2\sigma H$$

H curvature = 1/R
σ surface tension



Flow:
Drop deforms
→ nonuniform curvature
capillary stresses

area changes ⇒
extensible interface
constant tension

particle shape is not given a priori !

interfacial stresses ↔ hydrodynamic stresses

3

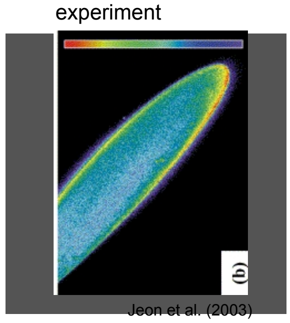
Soft particle: surfactant-covered drop

surfactant

Oil-like tail (hates water)

Polar head (likes water)

experiment



Jeon et al. (2003)

Equilibrium:
surfactant coverage
is uniform

normal forces=

$$2\sigma H$$

Flow:

shape deformation
→ *capillary stresses*

surfactant is redistributed
→ gradients in surface tension
Marangoni stresses

tangential forces= $-\nabla_s \sigma$

Surfactant effects on drop dynamics?

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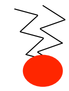
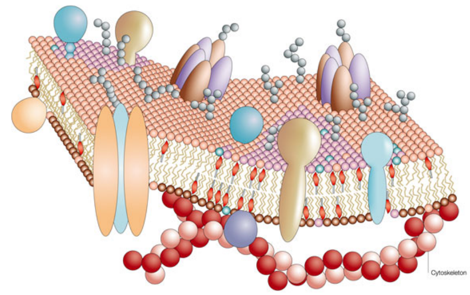
Soft particle: cell



Lipid bilayer:
main structural component of the cell m

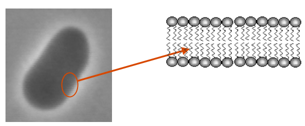


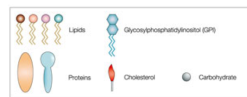
bilayer thickness ~ 5 nm
fluid, fixed number of lipids

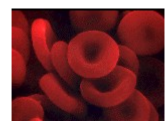
easier to bend than to stretch

$\kappa \approx 10\kappa_B T$ $K_a = 50\kappa_B T / nm^2$

vesicle:







$a \sim 10\text{-}100\ \mu m$
flickering

Simplest (mechanical) model of the RBC

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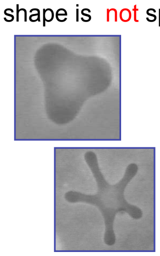
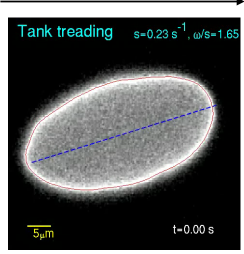
Soft particle: vesicle

Equilibrium:

shape is not spherical





Mader et al. EPJ (2006)
Steinberg's group PRL (2005, 2006, 2009) PNAS (2009)

Flow:

shape deforms:

capillary stresses

bending stresses

area constant ↔ *tension changes*

•isotropic tension
(global area constraint)

•variations along the surface
(local area constraint)

normal forces=

$$2\sigma H + \kappa\left[4H^3 - 4HK + 2\nabla_s^2 H\right]$$

tangential forces=

$$-\nabla_s \sigma$$

Bending and area constraint effects on vesicle dynamics?

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Outline

problem: dynamics of a soft particle in flow

a free-surface boundary problem

solutions:

surfactant-covered drop (surface tension) — capillary and Marangoni

vesicle (bending) — capillary, Marangoni and bending

examples

- emulsion rheology
- tumbling and tanktreading RBCs
- particle migration away from walls

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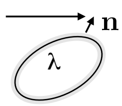
Problem formulation

soft particle in flow



$$\eta_i \nabla^2 \mathbf{u}_i - \nabla p_i = 0 \qquad i{=}in,\ out$$

nonlinearly coupled evolution equations of shape, tension, and flow field

- HD stresses discontinuous:

hydrodynamics stress balanced by *interfacial stresses*

$$\mathbf{t}[\mathbf{u}_{\mathrm{out}}] - \mathbf{t}[\mathbf{u}_{\mathrm{in}}] = \underbrace{2\sigma H\mathbf{n}}_{\text{capillary}} \underbrace{-\kappa\left(4H^3 - 4HK + 2\nabla_s^2 H\right)\mathbf{n}}_{\text{bending}} \underbrace{-\nabla_s \sigma}_{\text{Marangoni}}$$

capillary *bending* *Marangoni*

- velocity continuous across the interface

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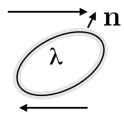
Problem formulation



shape evolution: interface moves with the fluid

$$\frac{\partial F}{\partial t} + \mathbf{u_s} \cdot \nabla F = 0 \qquad F = r - r_s$$

drop: *extensible interface*

(insoluble) surfactant:
surfactant conservation

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \Gamma \left(\nabla_s \cdot \mathbf{u}\right)$$

tension

$$\sigma\left(\Gamma\right) = \sigma_{\mathrm{eq}} + E\Gamma$$

σ_0 constant

vesicle: *incompressible membrane*

area conservation
(2D incompressibility)

$$\nabla_s \cdot \mathbf{u} = 0$$

tension

$$\sigma = \sigma_0 + \sigma_s$$

σ_s: variations along the surface (local area constraint)

σ_0: isotropic part (global area constraint)

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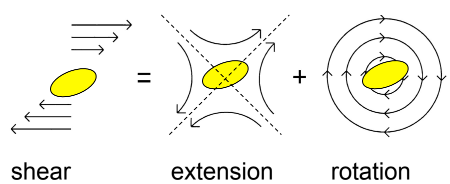
Particle dynamics: time scales



"distorting" $t_{\mathrm{d}} = \lambda \dot{\gamma}^{-1}$ convection by the extensional component of the flow

"restoring"

drop:

$t_{\mathrm{cap}} = \frac{\lambda \eta a}{\sigma}$ capillary relaxation

$t_{\mathrm{mar}} = \frac{\lambda \eta a}{\Delta \sigma}$ relaxation driven by interfacial tension gradients

vesicle:

$t_{\mathrm{ben}} = \frac{\lambda \eta a^3}{\kappa}$ bending relaxation

$t_{\mathrm{rot}} = \dot{\gamma}^{-1}$ rotation
(gets to be important at high λ)

$\lambda = \eta_{in}/\eta_{out} + 1$
viscosity contrast

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Dimensionless parameters

		drop	vesicle
flow strength $\frac{\text{relaxation}}{\text{distortion}}$	capillary number	$\frac{t_{cap}}{t_d} = Ca$	$\frac{t_{ben}}{t_d} = Ca$
flow-independent parameters		elasticity $E = \frac{t_{cap}}{t_{mar}}$	excess area Δ area - the area of a sphere with the same volume
rotation parameter		$\frac{t_{rot}}{t_d} = \lambda^{-1}$	

interplay of different time scales $\Rightarrow$ complex dispersion rheology

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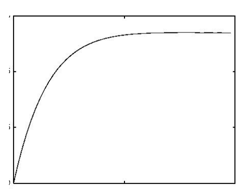
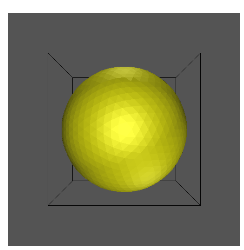
Example:

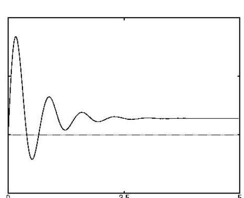
Drop deformation upon start-up of shear

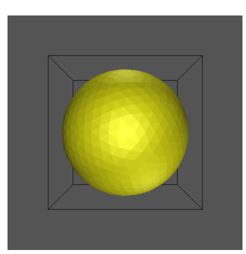
$t_{rot} \leftrightarrow t_{cap}$





$t_{rot} \gg t_{cap}$





$t_{rot} \ll t_{cap}$

shear stress

time t/t_{cap}

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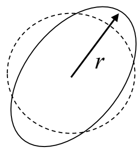
Problem solution

Analytical solutions: *Small deformations*:

perturbation expansions for small deviation from spherical shape

$$r(\theta,\phi) = 1 + \sum f_{jm} Y_{jm}(\theta,\phi)$$



θ, ϕ spherical coordinates

$$f \sim \varepsilon$$

Leading order shape evolution equation:

- *drop → linear*
- *vesicle → nonlinear* (area constraint)

$$\dot{f}_{jm} = \frac{\mathrm{i}m}{2} f_{jm} + h_{jm}(\lambda) + Ca^{-1}\Gamma(\sigma,\lambda,j) f_{jm} + O(\varepsilon^2)$$

rotation — forcing (flow) — response (interfacial stresses)

$$\varepsilon \equiv \{Ma, Ca, \lambda^{-1}, \Delta^{\frac{1}{2}}\} \ll 1$$

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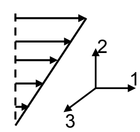
Problem solution: rheology

linear flow
$\mathbf{u} = \dot{\gamma}\mathbf{E}\cdot\mathbf{r}$



Effective stress for a dilute dispersion with particle volume fraction ϕ (only the traceless part)

$$\tau = \mu_0\left(2\dot{\gamma}\mathbf{E}^s\right) + \phi\tau^p$$

- suspension of *rigid* spheres
  Einstein (1905)

Newtonian $\tau^p = \frac{5}{2}\mu_0\left(2\dot{\gamma}\mathbf{E}^s\right)$

=> effective viscosity of the suspension

$$\mu^{eff} = \mu_0\left(1 + \frac{5}{2}\phi\right)$$

soft particles

- **shape deformation?**
- **surfactant?**
- **area constraint?**
- **bending rigidity ?**

Non-Newtonian ?

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Drops: emulsion rheology

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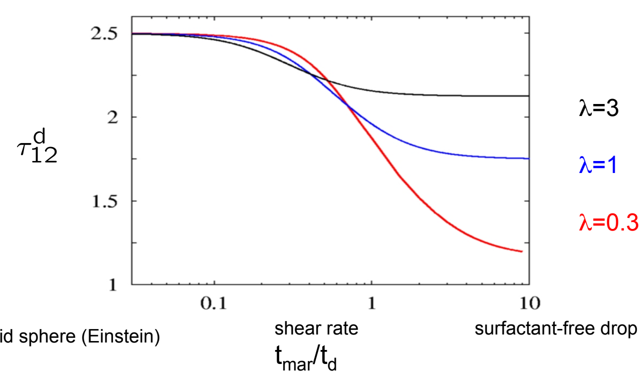
A simple example: emulsion of spherical surfactant-covered drops

$$\mu^{eff} = \mu_0 \left(1 + \phi \tau_{12}^d\right)$$

$t_d \leftrightarrow t_{mar}$

shear viscosity



Blawzdziewicz, Vlahovska, Loewenberg Physica A (2000)
Vlahovska, Blawzdziewicz, Loewenberg, J. Fluid Mech (2002)

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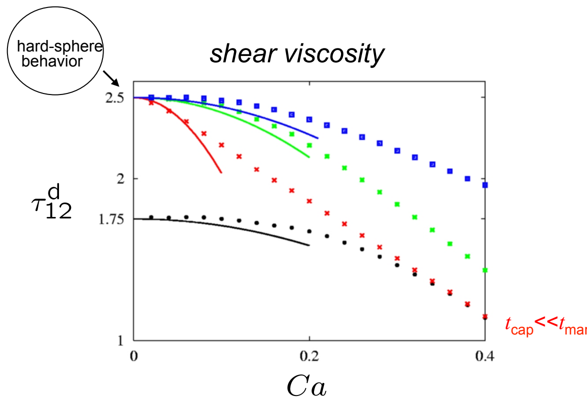
Deformable surfactant-covered drops

effect of surfactant elasticity

$t_{\mathrm{d}} \leftrightarrow t_{\mathrm{cap}} \leftrightarrow t_{\mathrm{mar}}$

viscosity contrast =1 $t_{\mathrm{d}} \sim t_{\mathrm{rot}}$

shear viscosity



shear rate t_{cap}/t_d

E=0.25 $t_{mar} > t_{cap}$
E=1 $t_{mar} \sim t_{cap}$
E=4 $t_{mar} < t_{cap}$
E=0 (surfactant-free drop)

symbols: BIM simulations
lines: small deformation theory

$$\tau_{12}^{\mathrm{d}} = \frac{5}{2} - Ca^2 C(E, \lambda)$$

Vlahovska, Blawzdziewicz, Loewenberg, Phys Fluids (2005), J. Fluid Mech (2009)

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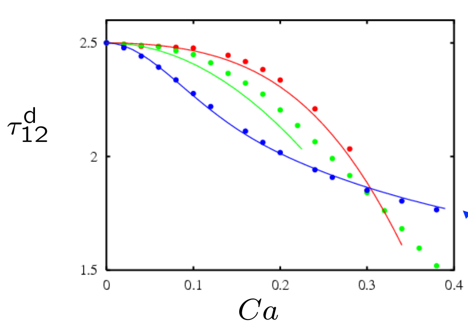
Deformable surfactant-covered drops

effect of viscosity contrast

$t_{\mathrm{d}} \leftrightarrow t_{\mathrm{rot}} \leftrightarrow t_{\mathrm{cap}}$

elasticity E=1 $t_{\mathrm{cap}} \sim t_{\mathrm{mar}}$

shear viscosity



λ=1 (bubble)
λ=2 (same fluid inside and outside)
λ=6

symbols: BIM simulations
lines: small deformation theory

$$\tau_{12}^{\mathrm{d}} = \frac{5}{2} - \lambda^{-1} C(Ca, E)$$

t_{rot}<<t_{cap}
main relaxation mechanism becomes rotation

clean drops with λ>4 do not break!

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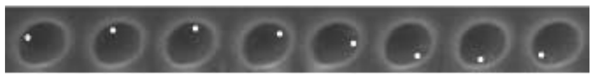
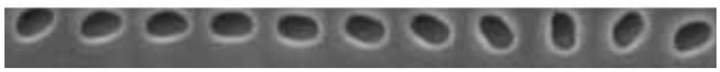
Vesicles: dynamics in shear flow

tank-treading



tumbling



Mader et al. 2006

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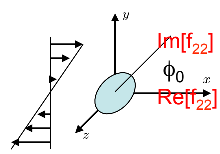
Vesicles:

shape evolution equation nonlinear



t>>t_ben shear flow: only j=2 mode survive

$$\dot{f}_{22} = \mathrm{i} f_{22} + \mathrm{i} h(\lambda) + 2h(\lambda)\Delta^{-1}\left(f_{22} - f_{2-2}\right) f_{22}$$

inclination angle:

$$\phi_o = \arctan\left(\frac{Im[f_{22}]}{Re[f_{22}]}\right)$$

independent of membrane properties!

Two fixed points:

Tank-treading $f_{22}, f_{2-2} \neq 0$ $\phi_0 \neq 0$

Saddle-node bifurcation

Tumbling $\phi_o = \arctan\sqrt{-1 + \frac{4h(\lambda)^2}{\Delta}}$ ➡ critical viscosity contrast

$h < \frac{\sqrt{\Delta}}{2} \quad \leftrightarrow \quad \lambda > \lambda_c$

Trembling $Re[f_{22}] = h(\lambda), \quad Im[f_{22}] = 0$ $\phi_0 = 0$

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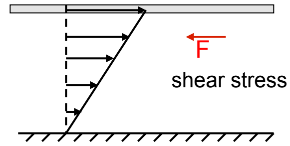
Misbah, PRL 2006; Vlahovska and Gracia, PRE 2007; Lebedev et al, NJP 2008

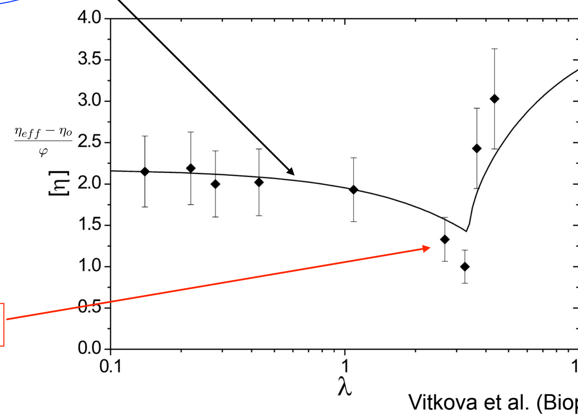
Effective viscosity of RBC suspension



$$\frac{5}{2} - \frac{\Delta(23\lambda + 32)}{16\pi}$$

Independent of shear rate

Einstein's result for a suspension of rigid spheres



tt-tb transition

note: volume fraction φ<10%

A signature microscopic dynamic on RBCs' suspension viscosity

Vitkova et al. (Biophys J 2008)

normal stresses

$$N_1 = -2N_2 = \sqrt{\frac{30}{\pi}}\Delta^{\frac{1}{2}}$$

(small excess area)

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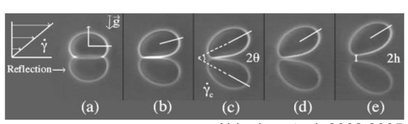
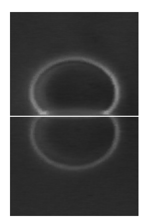
Wall effects: particle migration

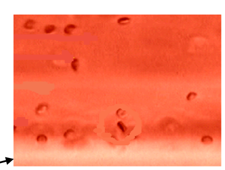


Abkarian et al. 2002,2005



Importance: Fahraeus effect

decrease in blood apparent viscosity



RBC depleted layer near the wall

(A. Viallat)

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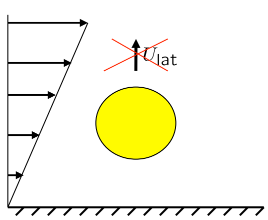
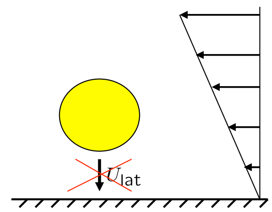
Wall effects: Particle migration in Stokes flow

spherical neutrally-buoyant particle does not lift !

Stokes flow equations $\eta \nabla^2 \mathbf{u} = \nabla p$ ⟶ linear





$\mathbf{u}$, $\mathbf{U}_{lat}$ reversing the flow ⟶ $-\mathbf{u}$, $-\mathbf{U}_{lat}$

migration is related to the normal stresses

Smart and Leighton (1991)

$$U_{lat} \sim \frac{1}{h^2}(N_1 - N_2)$$

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Problem solution

wall-bounded flow

$$\mathbf{u}(\mathbf{r}) = \mathbf{u}^{\infty} + \int \mathbf{G}(\mathbf{r},\mathbf{r}') \cdot \mathbf{t}(\mathbf{r}')\mathbf{dr}'$$

singularities formalism

↑ f

image system ↓ -f fluid (shear free surface)

stokeslet

↓↑ -2hf ⁚ $4h^2f$

stokeslet doublet source doublet rigid wall

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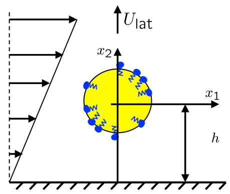
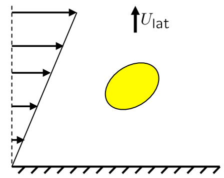
Particle migration

away from a wall in shear flows
towards the centerline in quadratic flows

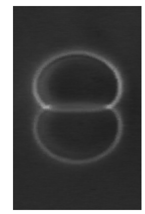
surfactant-covered spherical drop



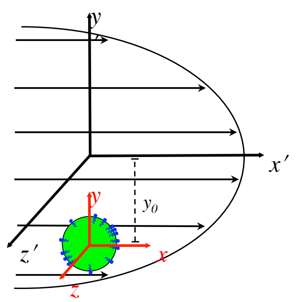
deformed drop



vesicle



Abkarian et al PRL



$$U_{\text{lat}} \sim \frac{\Delta^{\frac{1}{2}}}{h^2}$$

nonlinear effect

$$\longrightarrow \quad U_{\text{lat}} \sim \frac{Ca^2}{h^2}$$

Blawzdziwicz,Vlahovska, Loewenberg, Physica A, 2000
Hanna and Vlahovska, Phys Fluids 2010
Vlahovska and Gracia, PRE 2007

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Summary

microhydrodynamics of soft particles

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analytical theories: small deformations
numerical simulations: Boundary Integral Method

flow behavior of dispersions

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Other “soft” particles

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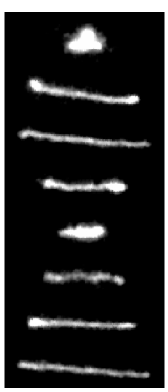
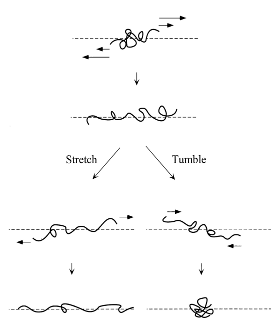
polymers

Coil—stretch transition of polymers in flow





Smith, Babcock and Chu, *Science* **283** p.1724 (1999)

importance:
drag reduction, von-Willebrand factor

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