

$$1. \text{ solenoidal vector: } \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} \text{ is solenoidal.}$$

$$2. \text{ irrotational vector: } \text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = 0$$

1. * Vector potential of a ~~conservative~~ magnetic field.

In electrodynamics one has that ~~$\vec{\nabla} \cdot \vec{B} = 0$~~

$\text{div } \vec{B} = \vec{\nabla} \cdot \vec{B} = 0$, where \vec{B} is the magnetic field. This implies that there exists another vector, \vec{A} , called "vector potential", such that $\vec{B} = \vec{\nabla} \times \vec{A} = \text{curl } \vec{A}$

-is the solution of $\vec{\nabla} \cdot \vec{B} = 0$.

Here let us show that $\vec{A} = \frac{1}{2} (\vec{B} \times \vec{r})$:

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \text{the vector } \vec{B} \text{ is solenoidal.}$

The solenoidal vector can be written as a curl of another vector, \vec{A} , called vector potential.

$$\begin{aligned} 2. \text{Curl } \vec{A} &= 2 \vec{\nabla} \times \vec{A} = 2 \vec{\nabla} \times \left(\frac{1}{2} \vec{B} \times \vec{r} \right) = \\ &= \vec{\nabla} \times (\vec{B} \times \vec{r}) = (\vec{\nabla} \cdot \vec{r}) \vec{B} - (\vec{B} \cdot \vec{\nabla}) \vec{r} = 3\vec{B} - \vec{B} = \vec{B}, \end{aligned}$$

where we used the fact that

$$\vec{A} \times (\vec{B} \times \vec{C}) = [\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})] \rightarrow$$

2. Gravitational & electrostatic forces are irrotational

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = 0$$

$$\vec{F} = C \cdot \frac{\vec{r}}{r^3} = C \cdot \frac{\vec{e}_r}{r^2}$$

Gravitational case: $C = -G m_1 m_2$

Coulomb law: $C = \frac{q_1 q_2}{4\pi\epsilon_0}$

Generally, if $\vec{\nabla} \times \vec{F} = 0$, \vec{F} is called conservative force.

$\Rightarrow \oint_S \vec{F} d\vec{r} = 0$ if S is simply connected.

If a force over a simply connected region S can be expressed as the negative gradient of a scalar function φ

$$\vec{F} = -\vec{\nabla} \varphi,$$

φ is called a scalar potential that describes the force by one function instead of three!

$$\oint_S \vec{F} d\vec{r} = \Rightarrow \vec{\nabla} \times \vec{F} = -\vec{\nabla} \times \vec{\nabla} \varphi = 0.$$

$\vec{F} \cdot d\vec{r}$

$$\oint \vec{F} \cdot d\vec{r} = - \oint \vec{\nabla} \varphi \cdot d\vec{r} = - \sum_{i=1}^3 \oint \frac{\partial \varphi}{\partial x_i} dx_i = - \oint \varphi d\varphi = 0$$

since the end points of a closed loop coincide.

Dirac Delta function.

Definition:

$$\delta(x) = 0 \quad \text{if } x \neq 0 \quad \text{and}$$

for "well" behaved f function (no

$$f(0) = \int_{-\infty}^{\infty} f(x) \delta(x) dx$$

$$\Downarrow \\ \int_{-\infty}^{\infty} \delta(x) dx = 1.$$

Problem: No such function exist,

in the usual sense of function.

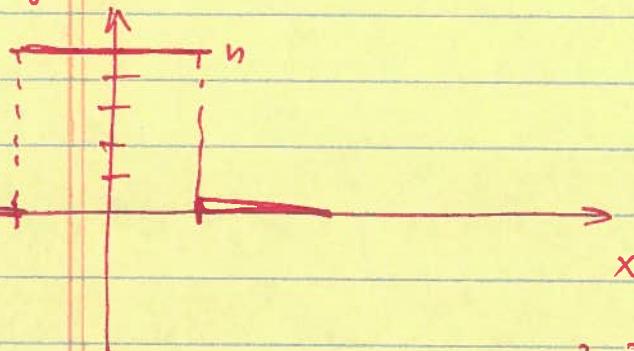
It can should be regarded ~~as a sequence of~~ ^{rigorously as a limit of} functions.

Exemplar:

1) δ

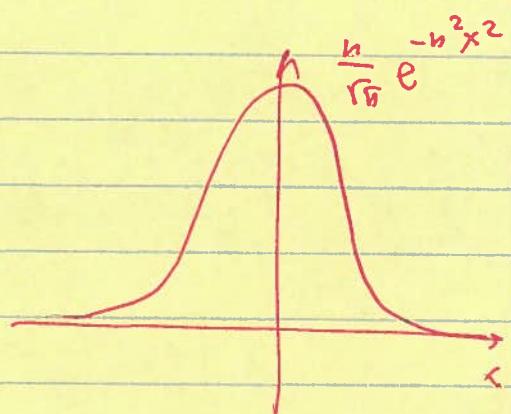
$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ n & -\frac{1}{2n} < x < \frac{1}{2n} \\ 0 & x > \frac{1}{2n} \end{cases}$$

$$y = \delta_n(x)$$



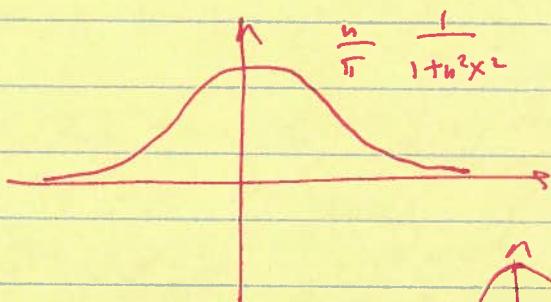
2)

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$$



3)

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{1+n^2 x^2}$$



4)

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2i} \int_{-n}^n e^{ixt} dt$$



=> we have a sequence of functions $\delta_n(x)$ such that.

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1.$$

Moreover, the sequence of integrals has no limit.

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0).$$

Then $\delta(x)$ is given by

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx$$

so that, $\delta(x)$ is mathematically a distribution (not a function) defined by the sequence $\delta_n(x)$.

* $\delta(x) = +\delta(x)$ is even in x .

* $\delta(ax) = \frac{1}{a} \delta(x), a > 0.$

↓

$$\int_{-\infty}^{\infty} f(x) \delta(ax) dx = \frac{1}{a} \int_{-\infty}^{\infty} f\left(\frac{y}{a}\right) \delta(y) dy = \\ = \frac{1}{a} f(0).$$

Integral representation for the Delta function.

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$

Now take

$$\delta_n(t-x) = \frac{\sin(n(t-x))}{\pi(t-x)} = \frac{1}{2\pi} \int_{-n}^n e^{i\omega(t-x)} d\omega,$$

then we have

$$f(x) = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(t) \delta_n(t-x) dt$$

δ_n is the sequence defining $\delta(x)$.

substitute $\delta_n(x)$ here:

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \int_{-n}^n e^{i\omega(t-x)} d\omega dt$$

— Interchanging order of integration and then taking the limit $n \rightarrow \infty$ we have

$$f(x) = \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega \right\} dt$$

Then the identification

$$\delta(t-x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega(t-x)} d\omega$$

is very useful representation of Dirac δ -function.