

Outline L5:

The most important examples of the equation of motion

- * Motion of a particle in a magnetic field
- * The harmonic oscillator.

Motion of a particle in a magnetic field:

q - charge of the particle

\vec{B} - is the magnetic field (uniform).

\vec{v} - velocity of the particle

\vec{a} - is the acceleration.

Then the magnetic force is:

$$\vec{F}_L = q \vec{v} \times \vec{B} = m \vec{a} \quad \text{- not a central force.}$$

Let us assume that \vec{B} is directed along the y -axis (we choose the coordinate system in this way). This means that

$$\vec{v} = \dot{x}(t) \vec{e}_x + \dot{y}(t) \vec{e}_y + \dot{z}(t) \vec{e}_z$$

$$\vec{a} = \ddot{x} \vec{e}_x + \ddot{y} \vec{e}_y + \ddot{z} \vec{e}_z$$

$$\vec{B} = B_0 \cdot \vec{e}_y.$$

Substitution of these vectors into our equation of state yields:

$$\begin{aligned} m(\ddot{x} \vec{e}_x + \ddot{y} \vec{e}_y + \ddot{z} \vec{e}_z) &= q(\dot{x} \vec{e}_x + \dot{y} \vec{e}_y + \dot{z} \vec{e}_z) \times (B_0 \vec{e}_y) \\ &= q B_0 (\dot{x} \vec{e}_z - \dot{z} \vec{e}_x) \end{aligned}$$

So our equations of motion become

$$(*) \left\{ \begin{array}{l} m \ddot{x} = -q B_0 \dot{z} \\ m \ddot{y} = 0 \\ m \ddot{z} = q B_0 \dot{x} \end{array} \right. \quad \text{How to solve the first and the third equations?}$$

The system is not separable. The second eq. of motion is easy to solve:

$$m \ddot{y}(t) = 0 \Rightarrow \dot{y} = v_{0y} = \text{const} \Rightarrow \boxed{y(t) = v_{0y} t + y_0}$$

where y_0 is also a constant.

To integrate the first and last equations we can use a trick:

define: $\alpha = \frac{q B_0}{m}$, so $\begin{cases} \ddot{x} = -\alpha \dot{z} \\ \ddot{z} = \alpha \dot{x} \end{cases}$

Differentiating the first eq yields

$$\dddot{x} = -\alpha \ddot{z} = -\alpha (\alpha \dot{x})$$

therefore

$$\dddot{x} = -\alpha^2 \dot{x} \quad \text{and}$$

$$\dddot{z} = -\alpha^2 \dot{z}$$

This is a set of two LINEAR, SECOND order differential equations with respect to functions $\dot{x}(t)$ and $\dot{z}(t)$!

Since these are linear, if ~~we~~ $x_1(t)$, ^{and} $x_2(t)$, (also $z_1(t)$ and $z_2(t)$) are solutions, then

$$A_1 x_1(t) + B_2 x_2(t)$$

$$A'_1 z_1(t) + B'_2 z_2(t)$$

are also solutions for any constant A, B and A', B' .

So all we need ~~to~~ to find are two special solutions. Then a linear combination, with 2 arbitrary coefficients, of these solutions will be the most general solution of our differential equations.

The two special solutions are

$$x_1(t) = \cos \alpha t$$

$$z_1(t) = \cos \alpha t$$

$$x_2(t) = \sin \alpha t$$

$$z_2(t) = \sin \alpha t$$

So that

$$\begin{cases} x(t) = A \cos \omega t + B \sin \omega t + x_0 \\ z(t) = A' \cos \omega t + B' \sin \omega t + z_0 \end{cases} \begin{array}{l} * \\ * \end{array}$$

with arbitrary A, B and A', B' are the most general

solutions of $\ddot{x} = -\omega^2 x, \ddot{z} = -\omega^2 z$.

A, A', B, B', x_0, z_0 are constants that are determined by the particle's initial position and velocity and by the equations of motion.

~~Ex 1~~ These solutions can be rewritten as

$$\begin{cases} (x - x_0) = A \cos \omega t + B \sin \omega t \\ (y - y_0) = v_0 t & (***) \\ (z - z_0) = A' \cos \omega t + B' \sin \omega t \end{cases}$$

The x and z coordinates are connected:

$$\begin{cases} \ddot{x} = -\omega^2 z \\ \ddot{z} = +\omega^2 x \end{cases} \left. \begin{array}{l} \text{Using } (***) \\ \text{here one obtains} \end{array} \right\} \text{back substitution}$$

from the first equation:

$$-d^2 A \cos dt - d^2 B \sin dt = -d(-dA' \sin dt + dB' \cos dt),$$

which holds for all t , in particular for $t=0$ and $(dt) = \frac{\pi}{2}$. Therefore equating coefficients in front of Cosines to each other, and those in front of Sines to each other, we obtain:

$$\begin{cases} -d^2 A = -d^2 B' \\ -d^2 B = d^2 A' \end{cases} \Rightarrow \begin{cases} A = B' \\ B = -A' \end{cases}$$

Therefore $(**)$ reduces to

$$\begin{cases} (x-x_0) = A \cos dt + B \sin dt \\ (y-y_0) = V_{0y} t \\ (z-z_0) = -B \cos dt + A \sin dt \end{cases} \quad \begin{cases} \cos'(dt) = -d \sin(dt) \\ \sin'(dt) = d \cos(dt) \end{cases}$$

IF at $t=0$ we had $V_z(t=0) = V_{0z}$ and $V_x(0) = 0$,

then we get from here

$$\begin{cases} V_x(t) = -Ad \sin(dt) + Bd \cos(dt) \\ V_z(t) = Bd \sin(dt) + Ad \cos(dt) \end{cases}$$

$$\text{Set } t \rightarrow 0 \Rightarrow \left. \begin{aligned} V_x(0) &= B\alpha = 0 \\ V_z(0) &= A\alpha = v_{0z} \end{aligned} \right\} \Rightarrow \begin{aligned} B &= 0 \\ A &= \frac{v_{0z}}{\alpha} \end{aligned}$$

$$S_0: \quad (x - x_0) = \frac{v_{0z}}{\alpha} \cos(\alpha t)$$

$$(y - y_0) = v_{0y} t$$

$$(z - z_0) = \frac{v_{0z}}{\alpha} \sin(\alpha t)$$

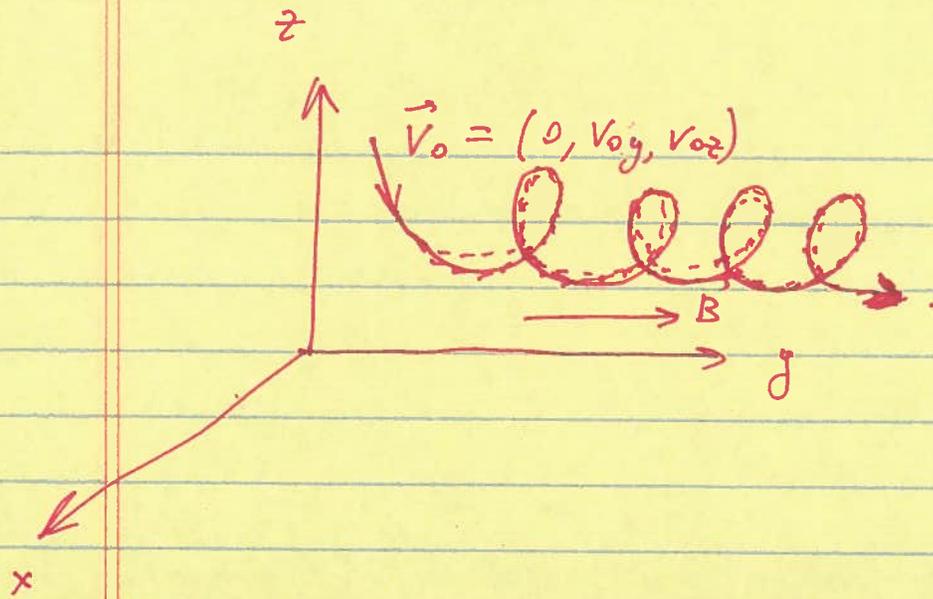
Finally using recalling that $\alpha = \frac{qB_0}{m}$, we

obtain

$$\left\{ \begin{aligned} x - x_0 &= \left(\frac{v_{0z} m}{q B_0} \right) \cos\left(\frac{q B_0}{m} t \right) \\ y - y_0 &= v_{0y} t \\ z - z_0 &= \left(\frac{v_{0z} m}{q B_0} \right) \sin\left(\frac{q B_0}{m} t \right) \end{aligned} \right.$$

These are parametric equations of a circular helix of radius $r_L = \frac{v_{0z} m}{q B_0}$ \div called Larmour radius.

along y-axis



The period of motion on the x - z plane is

$$T \cdot \frac{qB}{m} = 2\pi \Rightarrow T = 2\pi \left(\frac{m}{qB} \right), \text{ and}$$

does not depend on energy $\left(\frac{m v_{0y}^2}{2} + \frac{m v_{0z}^2}{2} \right)$, but only depends on the mass of the particle, its charge and strength of the magnetic field.

The angular frequency: $\omega_c = \frac{qB}{m}$ - is

called cyclotron frequency.

The Harmonic Oscillator

(in 1 Dimension)

Simplest nontrivial system where the force is a linear function of the position:

$$F(x) = -kx + F_0.$$

Equation of motion reads: $m\ddot{x}(t) = -kx(t) + F_0$

$$m\ddot{x} = -k\left(x - \frac{F_0}{k}\right).$$



From here we see that

- a) If $x > \frac{F_0}{k}$ and $k > 0$ then $F(x) < 0$ and the force brings the object to the left
- b) If $x < \frac{F_0}{k} \Rightarrow F(x) > 0$ and the force brings it to the right.

Finally, $x(t) = \frac{F_0}{k} = \text{const}$ is a solution of eq. of motion representing the equilibrium

It is a stable equilibrium, as we saw above. The force wants to decrease x if it is positive, and to increase x , if it is negative.

Notice, that if $k < 0$, then the equilibrium at $x = \frac{F_0}{k}$ is unstable.

It is helpful to shift the coordinates so that the ~~origin~~ equilibrium point is at the origin:

$$u = x - \frac{F_0}{k} \Rightarrow \dot{u} = \dot{x}, \ddot{u} = \ddot{x}$$

Eg. of motion thus acquires the form:

$$m\ddot{u} = -ku.$$

Or, redefining $u \rightarrow x$, the eq. of motion reduces to $m\ddot{x} = -kx$.

Again we have a linear second order differential equation as in previous problem. The general solution is

$$x(t) = A \cdot x_1(t) + B x_2(t), \text{ where}$$

$x_1(t)$ and $x_2(t)$ are two independent solutions. We know from previous problem that one can choose

$$x_1(t) = \cos(\omega t), \quad x_2(t) = \sin(\omega t)$$

where $\omega^2 = \frac{k}{m}$, $\omega = \sqrt{\frac{k}{m}}$ - called "angular frequency".

One can also write

$$x(t) = A \cos \omega t + B \sin \omega t = C \cos(\omega t + d)$$

$$\text{Set } t=0 \Rightarrow A = C \cdot \cos d$$

$$\omega t = \frac{\pi}{2} \Rightarrow B = C \cos\left(\frac{\pi}{2} + d\right) = -C \sin d$$

This gives us the relation between

A, B and C, d .

Initial conditions: Let $x(t=0) = A = X_0$

$$v(t=0) = 0 = \dot{x}(t=0) = B\omega = V_0$$

Therefore $x(t) = X_0 \cos \omega t + \frac{V_0}{\omega} \sin \omega t$

Dimensional analysis:

$\omega \cdot t$ is dimensionless variable \Rightarrow

$$\Rightarrow [\omega] = [\text{time}]^{-1}$$

Period of the motion is: $\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$

$T = \frac{1}{\nu}$ where ν is the "frequency" of the motion

$$\nu = \frac{\omega}{2\pi}$$