

## Outline: L5 Honors section

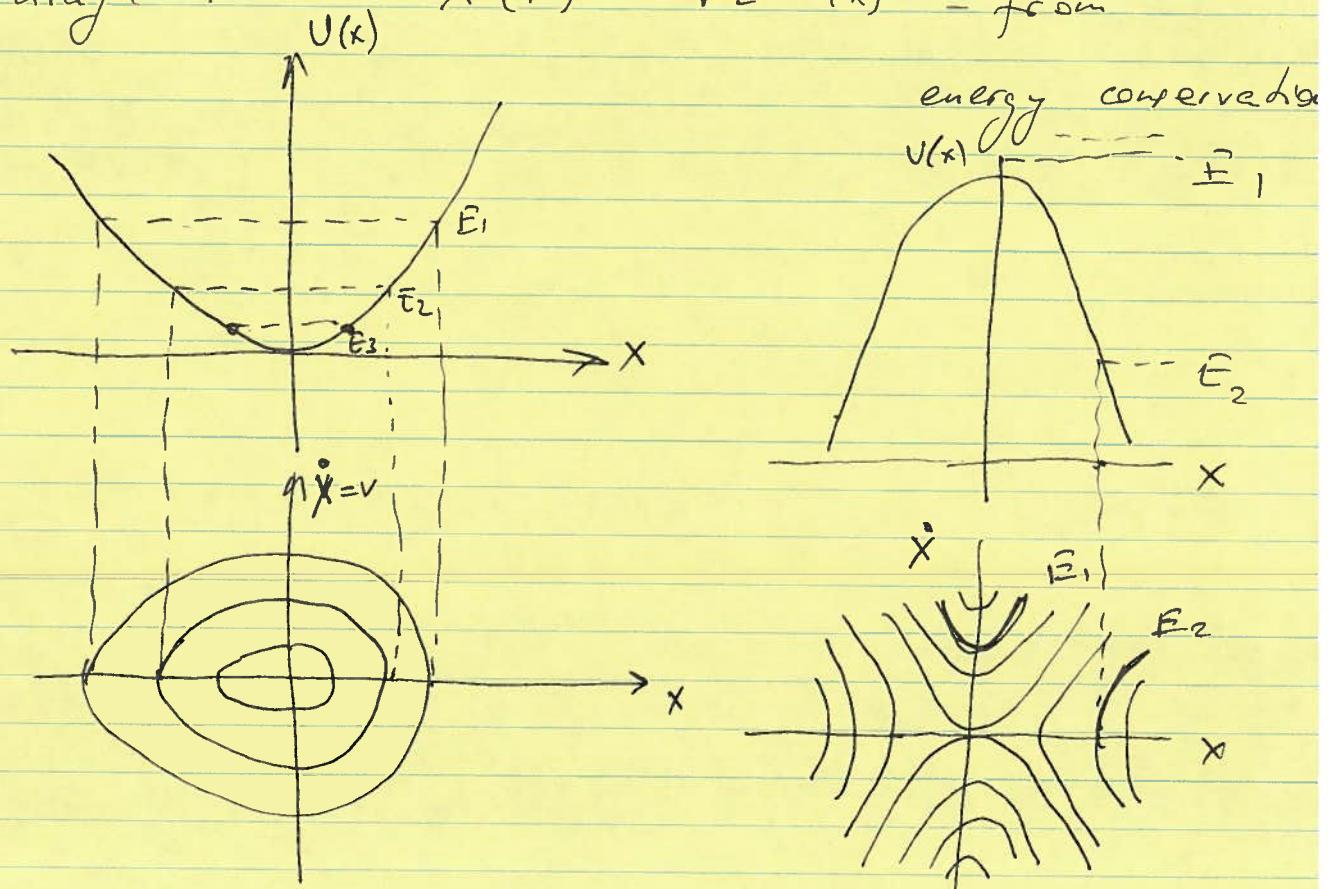
- \* Phase diagrams of nonlinear systems
  - self limiting equations/systems
- \* Chaos (in pendulum).

## Phase diagrams for nonlinear systems

Generally:  $m\ddot{x} + f(\dot{x}) + g(x) = h(t)$

- equation of motion of a damped and driven oscillator.

Phase diagram:  $\dot{x}(x) \sim \sqrt{E - U(x)}$  - from



For  $U(x) = \frac{1}{2}kx^2$  we get ellipses as.

$$\frac{mv^2}{2} + \frac{kx^2}{2} = E \quad \text{is an ellipse in } (x, v) \text{ plane.}$$

Having these in mind, one can construct the phase diagram for arbitrary potential  $U(x)$ .

Consider van der Pol equation (of motion) of nonlinear oscillations:

$$\ddot{x} + \mu(x^2 - a^2)\dot{x} + \omega_0^2 x = 0, \quad \mu > 0.$$

Consider 2 cases:

1. In the region where  $|x| > |a|$ , the coefficient of  $\dot{x}$  is positive  $\Rightarrow$  the system is damped.
2. If  $|x| < |a| \Rightarrow$  negative damping occurs.  $\Rightarrow$   $\Rightarrow$  amplitude of the motion increases.

Conclusion:

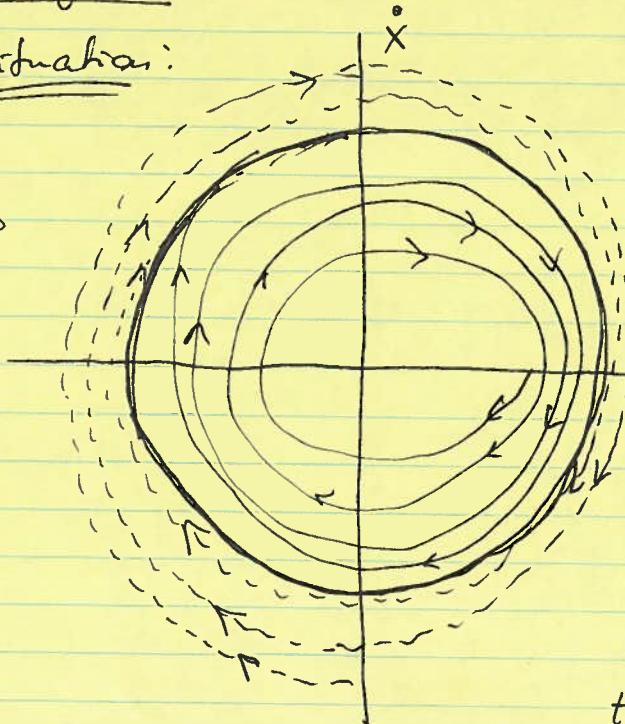
There must be some amplitude for which the motion is neither increases nor decreases with time.

Such a curve in the phase diagram is called the limit cycle.

Stable situation:

$$\mu \ll \omega_0^2 \Rightarrow$$

(for  $\mu=0, t$  we do not have such sinusoidal behavior for  $x(t), \dot{x}(t)$  anymore).



depending on

initial values  
of  $(x(t=0), \dot{x}(t=0))$

$\times$  one either  
can get the  
full line approaching  
the limit cycle or  
the dashed line.

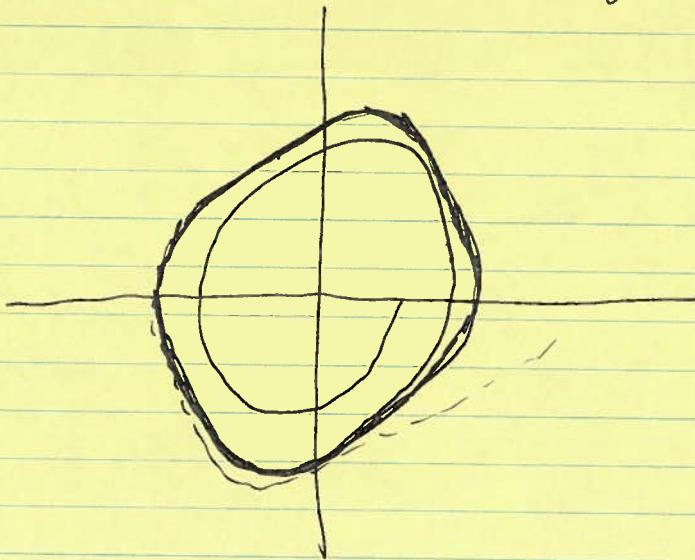
A system described by van der Pol's equation is self-limiting; that is, once set into motion under conditions that lead to increasing amplitude, the amplitude is automatically prevented from growing without bound. The system has this property whether the initial amplitude  $x(t=0)$  is  $>$  or  $<$  than the critical amplitude  $\alpha$ .

The solution for  $x$  of the circle is the result of choice for  $\alpha$  and  $\omega_0$ .

$$\text{If } \alpha = 1, \omega_0 = 1$$

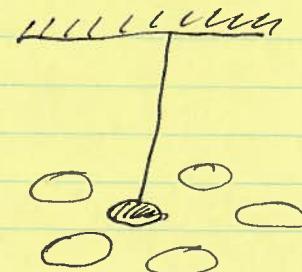
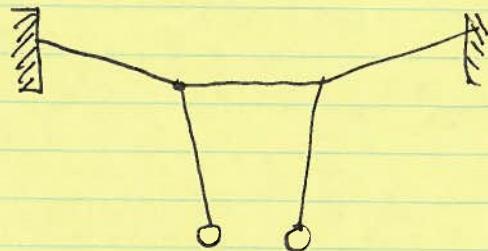
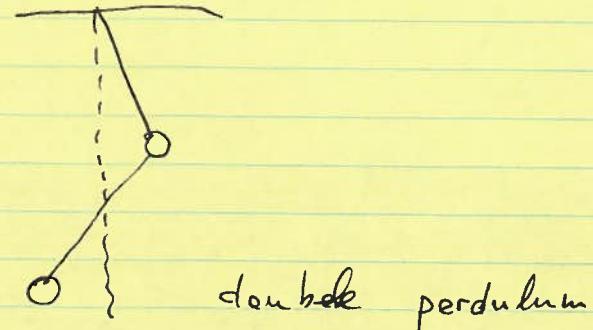
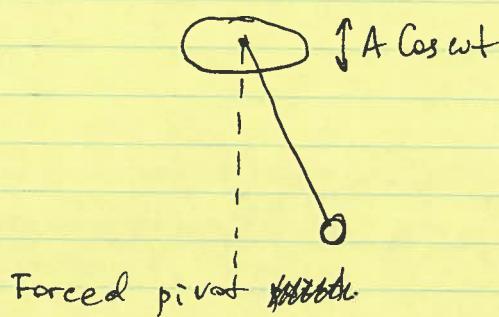
$$\Rightarrow \ddot{x} + \mu(x^2 - 1)x = 0. \quad \text{and we get } \sqrt{R_{\text{limit}}} = 2.$$

For  $\mu = 0.5$  the solution reaches the limit cycle much more quickly, and the limit cycle is distorted.



## Chaos in a Pendulum:

Consider damped and driven pendulum to introduce several chaos concepts.

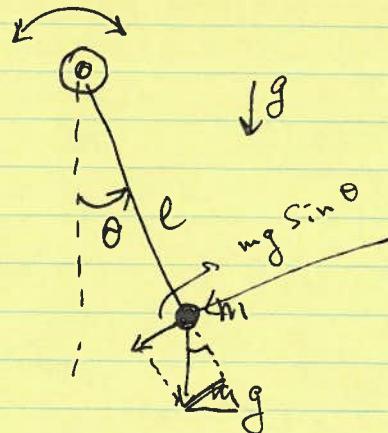


magnetic pendulum.

A damped pendulum is driven about its pivot point.

Forced motion  $\Rightarrow$  Torque =

$$M_d = -C \cos \omega t$$



apply force  
 $F(t) = F_0 \cos(\omega_0 t)$ .

The torque around the pivot point can be written as:

$$N = J \cdot \ddot{\theta} = -b \cdot \dot{\theta} - mgl \sin \theta + N_d \cdot \cos \omega_d t$$

↓                    ↓  
 moment of      angular  
 inertia          acceleration

damping coefficient.      angular velocity.  
 torque due to damping

$$= J = ml^2$$

$$\ddot{\theta} = -\frac{b}{ml^2} \dot{\theta} - \frac{g}{l} \sin \theta + \frac{N_d}{ml^2} \cos \omega_d t \quad (\omega_0^2 = g/l)$$

Introduce dimensionless variables  $t' = \omega_0 t$ , and

$\omega = \frac{\omega_d}{\omega_0}$ . The new dimensionless variables / parameters are:

$x = \theta$  - oscillating variable

$c = \frac{b}{ml^2 \omega_0}$  - damping coefficient

$F = \frac{N_d}{ml^2 \omega_0^2} = \frac{N_d}{mgl}$  - driving force strength.

$t' = \frac{t \omega_0}{\omega_0} = \sqrt{\frac{g}{l}} \cdot t$  - dimensionless time

$\omega = \frac{\omega_d}{\omega_0} = \sqrt{\frac{l}{g}} \cdot \omega_d$  - driving angular frequency.

Note that

$$\dot{x} = \frac{dx}{dt'} = \frac{d\theta}{dt} \cdot \frac{dt}{dt'} = \frac{d\theta}{dt} \cdot \frac{1}{\omega_0}$$

$$\ddot{x} = \frac{d^2x}{dt'^2} = \frac{d^2\theta}{dt^2} \left( \frac{dt}{dt'} \right)^2 = \frac{d^2\theta}{dt^2} \cdot \frac{1}{\omega_0^2} = \frac{\ddot{\theta}}{\omega_0^2}.$$

Using these dimensionless variables we get for EM:

$$\ddot{x} = -c\dot{x} - \sin x + F \cos \omega t'.$$

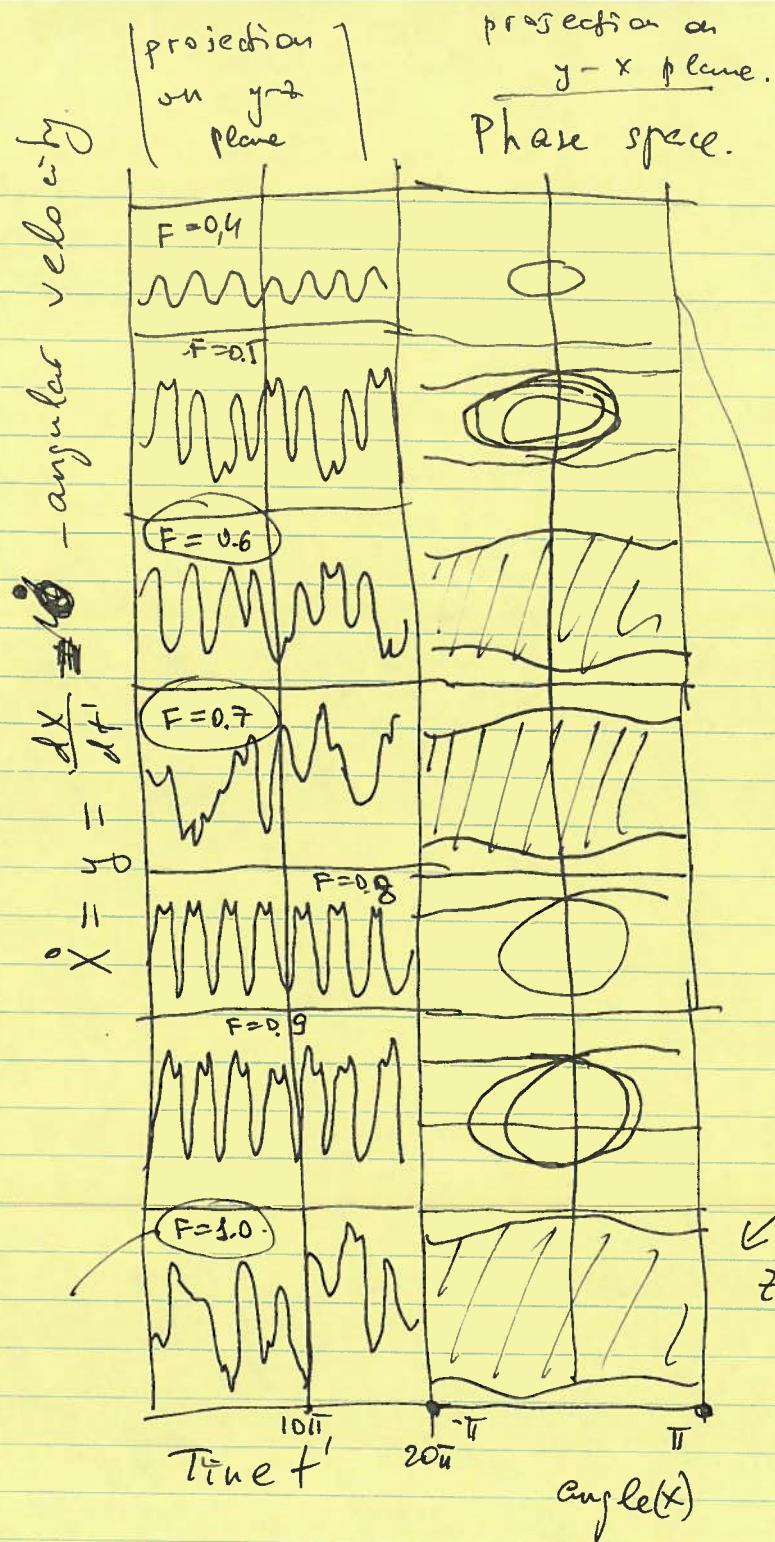
- can be solved numerically for  $x(t)$ ,  $\dot{x}(t)$  for given sets of parameters  $\{c, F, \omega\}$ .

The second order & non-linear diff. equation can be reduced to a set of 2 first-order equations:

$$y = \frac{dx}{dt'}$$

$$\frac{dy}{dt'} = -cy - \sin x + F \cos z$$

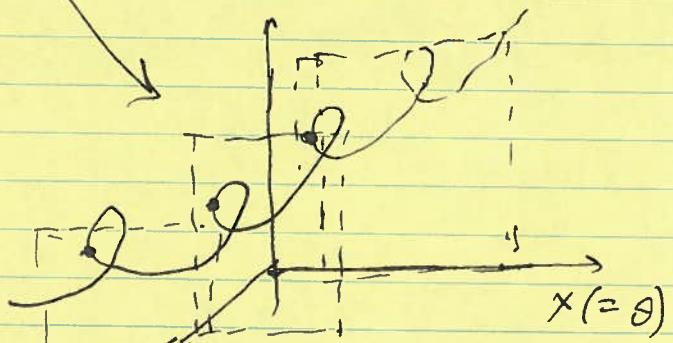
where  $z = \omega t'$ .



$$y = \dot{x} = \dot{y} \quad \text{vs } x (= \theta) \\ z (= \omega t')$$

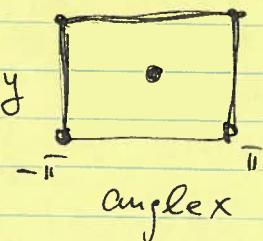
gives a 3D phase diagram

$$y (= \frac{dx}{dt} = \dot{\theta})$$



3D phase  
diagram =  
= Poincaré plot.

Poincaré section



For  $F = 0.6, 0.7, \text{ and } 3.0$

the motion is chaotic!