

Outline of L1:

- * Concepts of a scalar and a vector
- + coordinate transformations in 2D and 3D.
- * rotation matrices and their properties.

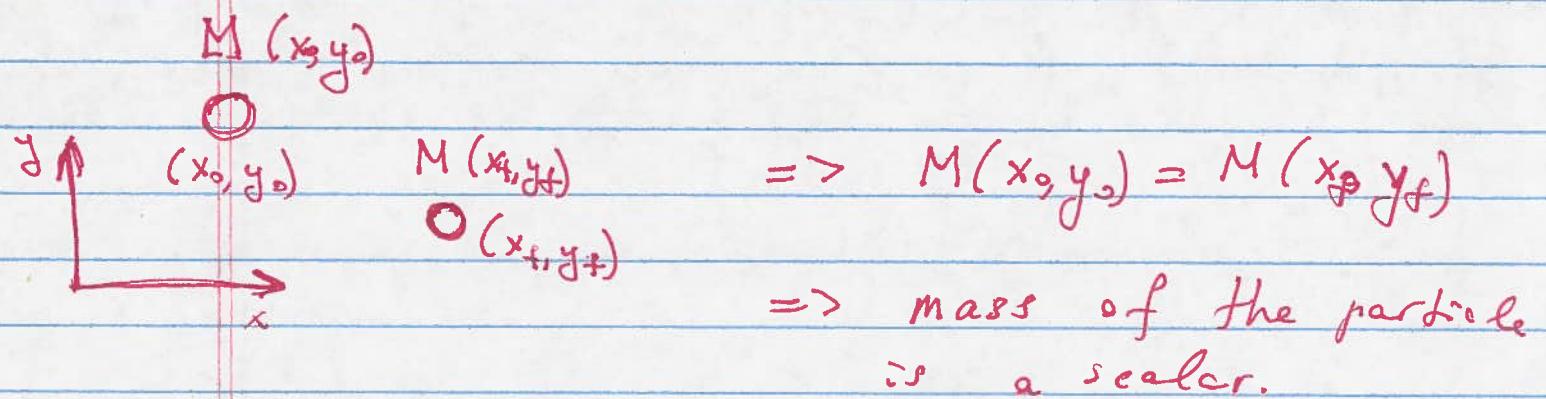
Basic math. concepts: linear / coordinate algebra.

(1)

* Concept of a Scalar.

Quantities that are invariant under coordinate transformations are termed scalars.

Example: The mass of the particle is a scalar.



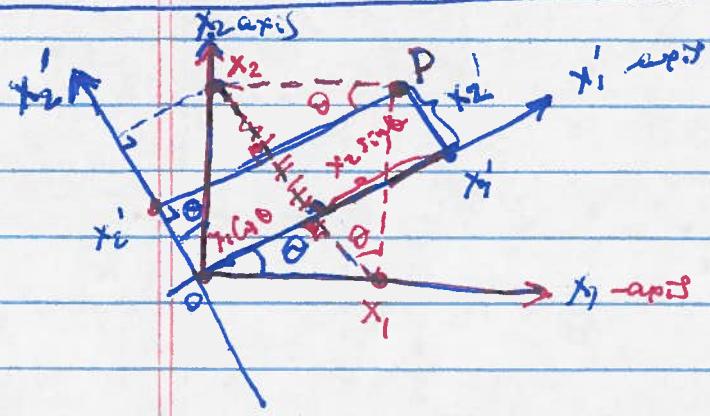
Other examples: temperature, speed, etc.

~~Properties:~~

On the other hand: { direction of motion /
{ direction of the force /

etc cannot be specified in such a manner.

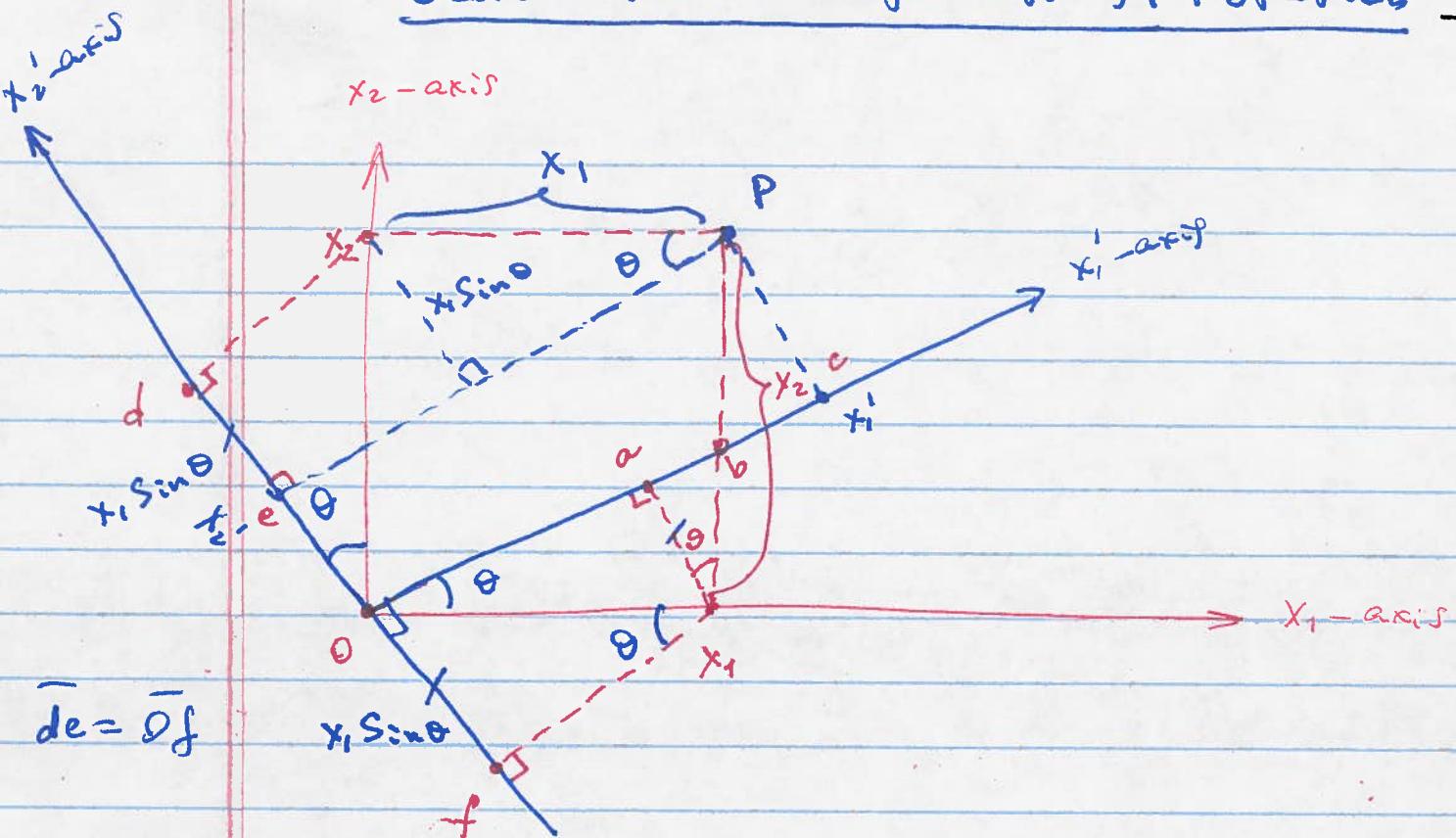
These are vectors.



$$x'_1 = x_1 \cos \theta + x_2 \sin \theta = \\ = x_1 (\cos \theta + x_2 \cos(\frac{\pi}{2} - \theta))$$

$$x'_2 =$$

Coordinate transformations: rotations - 2 -



The position of P can be represented in two coordinate systems, one rotated from the other.

Initial system $P(x_1, x_2, x_3)$

Final system $P(x'_1, x'_2, x'_3)$

Under rotation, coordinates transform as

$$x'_1 = da + ab + bc = \underbrace{x_1 \cos \theta + x_2 \sin \theta}_{= 0} = \\ = x_1 \cos \theta + x_2 \cos\left(\frac{\pi}{2} - \theta\right)$$

$$x'_2 = \overline{d}d - \overline{d}e = \overline{d}d - \overline{d}f =$$

$$= -x_1 \sin \theta + x_2 \cos \theta = x_1 \cos\left(\frac{\pi}{2} + \theta\right) + x_2 \cos \theta$$

Now we introduce the following notations: we write
the angle θ between x'_i -axis and x_i -axis

$$\theta = \angle x'_i, x_i = (x'_i, x_i) \quad \text{#}$$

In general we write $\angle x'_i, x_j = (x'_i, x_j)$.

Furthermore, we define a set of numbers

$$! \quad \lambda_{ij} = \cos(x'_i, x_j) \Rightarrow$$

called direction cosine of the x'_i axis relative to the x_j axis

\Rightarrow For our 2D rotation we have

$$\lambda_{11} = \cos(x'_1, x_1) = \cos \theta$$

$$\lambda_{12} = \cos(x'_1, x_2) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\lambda_{21} = \cos(x'_2, x_1) = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\lambda_{22} = \cos(x'_2, x_2) = \cos \theta$$

\Rightarrow equations of transformation now become

$$x'_1 = x_1 \cos(x'_1, x_1) + x_2 \cos(x'_1, x_2) = \lambda_{11} x_1 + \lambda_{12} x_2$$

$$x'_2 = x_1 \cos(x'_2, x_1) + x_2 \cos(x'_2, x_2) = \lambda_{21} x_1 + \lambda_{22} x_2$$

$$x'_3 = x_3 \rightsquigarrow$$

In general, in 3 dimensions we have

$$\left\{ \begin{array}{l} x_1' = \lambda_{11}x_1 + \lambda_{12}x_2 + \lambda_{13}x_3 \\ x_2' = \lambda_{21}x_1 + \lambda_{22}x_2 + \lambda_{23}x_3 \\ x_3' = \lambda_{31}x_1 + \lambda_{32}x_2 + \lambda_{33}x_3 \end{array} \right.$$

or, in summation notation: $x_i' = \sum_{j=1}^3 \lambda_{ij}x_j$,
 $i = 1, 2, 3$.

The inverse transformation is

$$x_i = \sum_{j=1}^3 \lambda_{ji}x_j', \quad i=1, 2, 3.$$

Finally, if one writes a coordinate of P
 a) a column / row

$$\overrightarrow{P}(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \vec{P}(x_1, x_2, x_3) = (x_1, x_2, x_3),$$

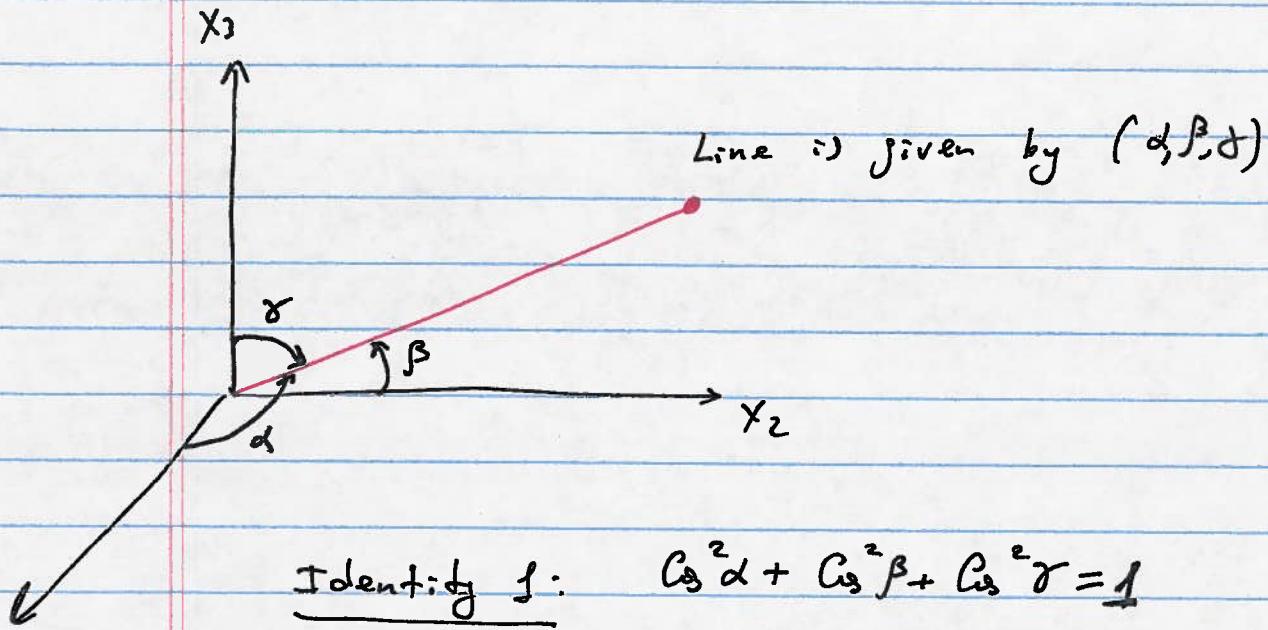
then the desired coordinate transformation will be a ~~A-matrix~~

recall also:

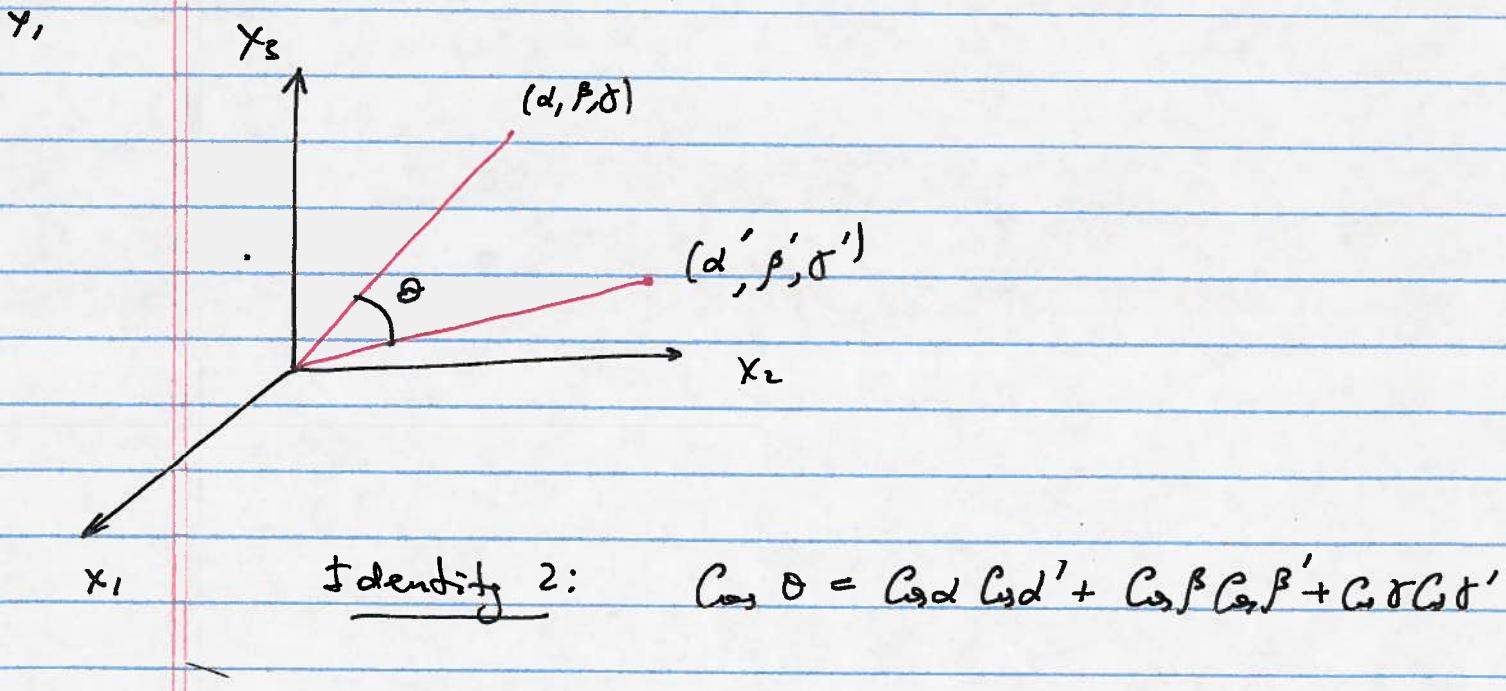
$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}}_{\text{A-matrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Properties of rotation matrices:

Geometric identities:



$$\text{Identity 1: } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$



$$\cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'$$

Homework (will be assigned on Thursday):

using these identities show that

$$(1) \quad \sum_{j=1}^3 \lambda_{ij} \lambda_{kj} = 0, \quad k \neq j.$$

$$(\text{i.e. } \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = 0 \Rightarrow \tilde{\lambda}_2 = 0)$$

$$(2) \quad \sum_{j=1}^3 \lambda_{ij} \lambda_{ij} = 1, \quad i = k$$

$$(\text{i.e. } \lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1).$$

One may combine these relations into one

$$\boxed{\sum_{j=1}^3 \lambda_{ij} \lambda_{kj} = \delta_{ik}},$$

where $\delta_{ik} = \begin{cases} 0, & \text{if } i \neq k \\ 1, & \text{if } i = k \end{cases}$ is called

Kronecker delta symbol.

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Please check

The validity of this eq. depends on the coordinate axes in each of the systems being mutually perpendicular

\Rightarrow orthogonal systems.

Returning to our matrix notations:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \Delta \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix}, \text{ where } (\Delta)_{ij} = \lambda_{ii}$$

~~Def~~
A transposed matrix is a matrix derived from an original matrix by interchange of rows and columns.

$$(\Delta^T)_{ij} = (\Delta)_{ji} \quad \text{or}$$

$$\lambda_{ij}^T = \lambda_{ji}.$$

We have that $(\Delta^T)^T = \Delta$.

Identity matrix is when multiplied by another matrix, leaves the latter unaffected.

$$\mathbf{1} \cdot \Delta = \Delta, \quad \Delta \cdot \mathbf{1} = \Delta.$$

$$\mathbf{1} \cdot \Delta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \Delta.$$

Orthogonal systems and their rotation matrices:

Consider orthogonal $\overset{\text{rotation}}{\text{matrix}}$ Δ in \mathbb{R}^2 :

$$\Delta = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \quad \text{then}$$

$$\Delta \cdot \Delta^T = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \cdot \begin{pmatrix} \lambda_{11} & \lambda_{21} \\ \lambda_{12} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \lambda_{11}^2 + \lambda_{12}^2, & \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} \\ \lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12}, & \lambda_{21}^2 + \lambda_{22}^2 \end{pmatrix}$$

Using the orthogonality relation we find

$$\lambda_{11}^2 + \lambda_{12}^2 = \lambda_{21}^2 + \lambda_{22}^2 = 1$$

$$\lambda_{21}\lambda_{11} + \lambda_{22}\lambda_{12} = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} = 0$$

$$\Rightarrow \boxed{\Delta \cdot \Delta^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}}.$$

On the other hand, the inverse matrix is defined as:

$$\boxed{\Delta \Delta^{-1} = \mathbb{I}} \rightarrow \text{identity matrix}$$

Comparing these equations we see

$$\boxed{\Delta^T = \Delta^{-1}}$$

for orthogonal matrices.

Matrix algebra:

1. Matrix multiplication is not commutative in general

$$\Delta_1 \cdot \Delta_2 \neq \Delta_2 \cdot \Delta_1$$

* The special case of multiplication of Δ and Δ^{-1} is commutative

$$\Delta \cdot \Delta^{-1} = \Delta^{-1} \cdot \Delta = 1.$$

* The identity matrix commutes: $\Delta \cdot 1 = 1 \cdot \Delta = \Delta$

2. Associativity: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

3. Addition of matrices: $\Delta = \Delta_1 + \Delta_2$ means

$$(\Delta)_{ij} = (\Delta_1)_{ij} + (\Delta_2)_{ij}. -$$

- defined if Δ_1 and Δ_2 have the same dimensions.

Definition of a scalar and a vector

Coordinate transform: $x_i' = \sum_{j=1}^3 \lambda_{ij}; x_j \text{ with } \sum_{j=1}^3 \lambda_{ij} \lambda_{kj} = \delta_{ik}$

a) if quantity ϕ is unaffected, i.e. $\phi(x_1, x_2, x_3) = \phi(x_1', x_2', x_3')$ then it is called a scalar.

b) if $\vec{A} = (A_1, A_2, A_3)$ transforms as $A_i' = \sum_{j=1}^3 \lambda_{ij} A_j$, then \vec{A}' is called a vector.