

## Lecture 10.

Outlook:

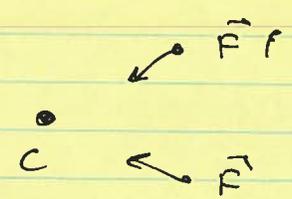
- \* Central-Force motion
- \* Effective potential
- \* Planetary motion - Kepler's problem

### Central force and central potential

As we discussed in Lecture 3, a central force between two particles A and B,  $\vec{F}_{AB}$  is given by  $\vec{F}_{AB} = (\vec{r}_A - \vec{r}_B) \cdot \varphi_{AB}(|\vec{r}_A - \vec{r}_B|)$ . Here  $\varphi$  is a scalar function of distance  $|\vec{r}_A - \vec{r}_B|$ .

Suppose we have a center, C, that exerts force  $\boxed{\vec{F} = \vec{F}(r) \vec{e}_r}$  to a particle residing at point  $\vec{r}$  (assume C is at the origin of the coordinate system).  $r = |\vec{r}|$  is the distance between them  $\Rightarrow \Rightarrow$  we have a central force  $\vec{F}(r)$ .

In polar coordinates we have that  $\vec{F}(r, \theta, \varphi) = F(r) \cdot \vec{e}_r$



As we know from Lect. 3, the central force is conservative. The corresponding potential  $U(r)$  can be found from  $F(r) = - \frac{dU(r)}{dr}$ .

Important property: If the potential  $U = U(r)$ , then the motion takes place on a plane.

Proof:  $\forall U = U(r) \Rightarrow \vec{F} = - \frac{dU}{dr} \cdot \vec{e}_r$  and

thus the torque  $\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \left( \frac{dU}{dr} \cdot \vec{e}_r \right) = 0$ .  
 Therefore  $\vec{\tau} = \frac{d\vec{l}}{dt} = 0$  and thus

the angular momentum  $\vec{l} = \vec{r} \times \vec{p}$  is a fixed vector in space  $\vec{l} = \vec{l}_0$ . From here we see that  $\vec{r}$  will always be on the plane  $\perp$  to  $\vec{l}$ .

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Since motion is planar, we study it in polar coordinates  $(r, \theta)$ , without worrying the third coordinate.

In polar coordinates we have that

$$\vec{a} = \underbrace{\left( \ddot{r}(t) - r \dot{\theta}^2 \right)}_{\text{radial component}} \vec{e}_r + \underbrace{\left( r \ddot{\theta} + 2 \dot{r} \dot{\theta} \right)}_{\text{angular component}} \vec{e}_\theta.$$

Since in the case of central force  $\vec{F} = - \frac{dU}{dr} \cdot \vec{e}_r$ , Newton's equations of motion acquire the form

$$\begin{cases} m (\ddot{r} - r \dot{\theta}^2) = - \frac{dU}{dr} & \rightarrow \text{radial components} \\ r \ddot{\theta} + 2 \dot{r} \dot{\theta} = 0 & \rightarrow \text{for angular components.} \end{cases}$$

The angular comp part of EM can be readily integrated since

$$r \ddot{\theta} + 2 \dot{r} \dot{\theta} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \Rightarrow \frac{d}{dt} (r^2 \dot{\theta}) = 0$$

and thus  $r^2 \dot{\theta} = \text{constant}$ .

Reminder: Conservation of  $r^2 \cdot \dot{\theta} = \text{const}$  means that the absolute value  $|\vec{l}'| = l_0 = m r^2 \dot{\theta}$  is conserved.

$$|\vec{l}'| = |\vec{r} \times \vec{p}'| = r \cdot p \cdot \sin(\hat{p}' \hat{r}) = r \cdot p_{\theta} = l_0 = m r^2 \dot{\theta}$$

Since  $p_{\theta} = m r \dot{\theta}$  is the angular component.

So one can express  $\dot{\theta}$  in

terms of the conserved  $|\vec{l}'| = l$  as follows:

$$\dot{\theta} = \frac{l_0}{m r^2} \quad \text{Therefore, the radial equation of}$$

motion  $m(\ddot{r} - r\dot{\theta}^2) = -\frac{dU}{dr}$  takes the form:

$$m\ddot{r} - m r \left( \frac{l_0}{m r^2} \right)^2 = -\frac{dU}{dr} \Rightarrow$$

$$\Rightarrow \boxed{m\ddot{r} = -\frac{dU}{dr} + \frac{l_0^2}{m r^3}} \quad (*)$$

One can now define an effective potential

$$V_{\text{eff}}(r) = U(r) + \frac{l^2}{2mr^2} \quad \text{so that our}$$

equation of motion (\*) can be interpreted as an effective 1D motion

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr}$$

This equation now can be integrated by using the trick:

$$\frac{m \dot{r}^2}{2} + V_{\text{eff}}(r) = E = \text{const.}$$

Note however that  $V_{\text{eff}}$  depends on  $l$ , which is constant.

To study the motion in central potentials, one has to

- |                             |                                               |
|-----------------------------|-----------------------------------------------|
| 1. plot $V_{\text{eff}}(r)$ | 2. analyze the motion for fixed $E$ and $l$ . |
|-----------------------------|-----------------------------------------------|

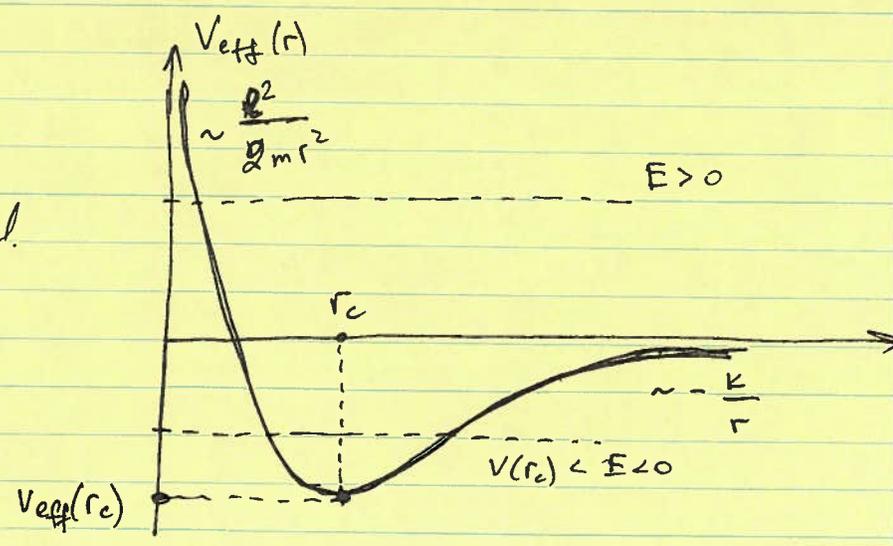
Note that  $E$  must be s.t.  $E > V_{\text{eff}}$  for motion to take place.

For the case of inverse-square-law central-force motion  $F(r) = -\frac{k}{r^2} \Rightarrow U(r) = -\int F(r)dr = -\frac{k}{r}$ , and

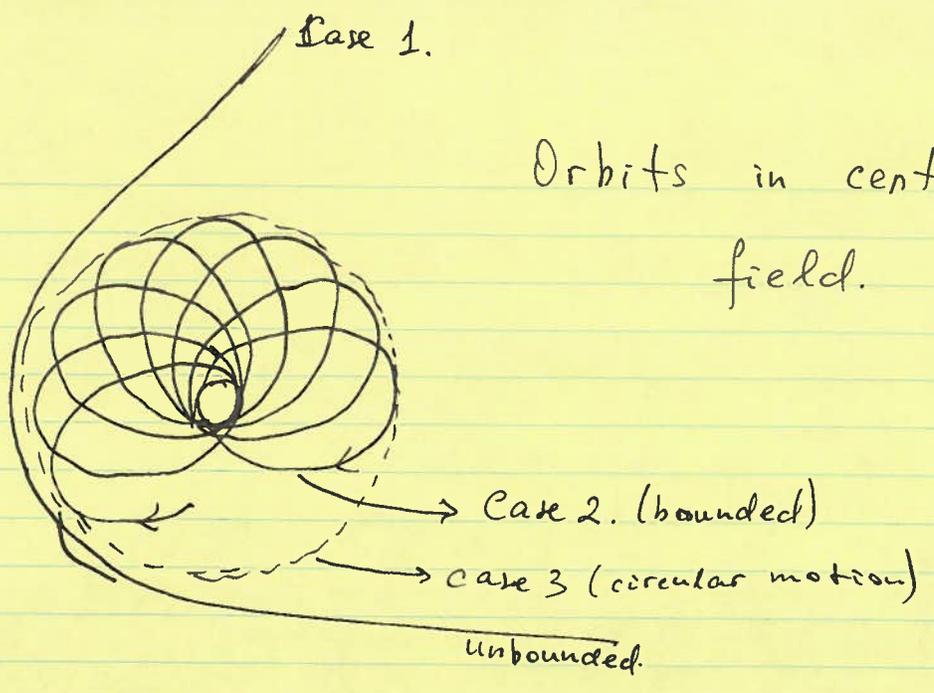
$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{l^2}{2mr^2}, \text{ where } l \neq 0 \text{ is fixed.}$$

Possible cases:

- $E \geq 0 \Rightarrow$  motion is unbounded.
- $V(r_c) < E < 0$ : Bounded
- $E = V(r_c)$ : circular motion.



# Orbits in central field.



## Planetary motion - Kepler's problem

To study the shape of the orbits, we do not care about the time dependence. We want to know the function  $r(\theta)$ !

We have that

$$\frac{dr}{dt} = \frac{dr}{dt} \cdot \frac{d\theta}{dt} = \frac{dr}{d\theta} \cdot \dot{\theta} = \frac{dr}{d\theta} \cdot \left( \frac{l}{mr^2} \right)$$

chain rule used

$$\begin{aligned} \frac{d^2r}{dt^2} &= \frac{d}{dt} \left[ \frac{dr}{d\theta} \cdot \left( \frac{l}{mr^2} \right) \right] = \frac{d\theta}{dt} \cdot \frac{d}{d\theta} \left[ \frac{l}{mr^2} \cdot \frac{dr}{d\theta} \right] = \\ &= \frac{l}{mr^2} \left[ \frac{l}{m} \right] \frac{d}{d\theta} \left( \frac{1}{r^2} \frac{dr}{d\theta} \right) = \dots \end{aligned}$$

Substituting all into our equation of motion one obtains:

$$(**) \quad \frac{d^2 r}{d\theta^2} - \frac{2}{r} \left( \frac{dr}{d\theta} \right)^2 - r = \frac{m r^4}{e^2} F(r), \quad \text{where} \\ F(r) = - \frac{dU(r)}{dr}.$$

To solve this equation, one can introduce a new variable:

$$x = \frac{1}{r} \Rightarrow \frac{dr}{d\theta} = \frac{dx}{d\theta} \frac{dr}{dx}, \quad \frac{d^2 r}{d\theta^2} = -\frac{1}{x^2} \frac{d^2 x}{d\theta^2} + \frac{2}{x^3} \left( \frac{dx}{d\theta} \right)^2$$

so that substitution into Eq (\*\*) yields

$$\frac{1}{x^2} \frac{d^2 x}{d\theta^2} + \frac{2}{x^3} \left( \frac{dx}{d\theta} \right)^2 - \frac{2x}{x^4} \left( \frac{dx}{d\theta} \right)^2 - \frac{1}{x} = \frac{m}{e^2 x^4} F\left(\frac{1}{x}\right)$$

Therefore:

$$\boxed{\frac{d^2 x}{d\theta^2} + x = - \frac{m}{e^2 x^2} F\left(\frac{1}{x}\right)}$$

This is a one-dimensional equation of a harmonic oscillator with a driving force  $-\frac{m}{e^2 x^2} F(1/x)!$

In case of the Kepler problem:  $F(r) = -\frac{k}{r^2} = -kx^2$

$\Rightarrow$  the equation acquires a simple form:

$$\boxed{\frac{d^2 x}{d\theta^2} + x = \frac{mk}{e^2}}$$

The solution is the same as the harmonic oscillator (note however the role of  $t$  is given to  $\theta: t \rightarrow \theta$ ).

$$r(\theta) = \underbrace{\frac{mk}{l^2}}_{\substack{\text{particular} \\ \text{sol. of eq.}}} + \underbrace{A \cos(\theta - \theta_0)}_{\substack{\text{general sol} \\ \text{of homogeneous} \\ \text{eq.}}} = \frac{mk}{l^2} \left[ 1 - \varepsilon \cos(\theta - \theta_0) \right]$$

called  
ECCENTRICITY.

The parameter  $\varepsilon$  (eccentricity) is related to energy and angular momentum through

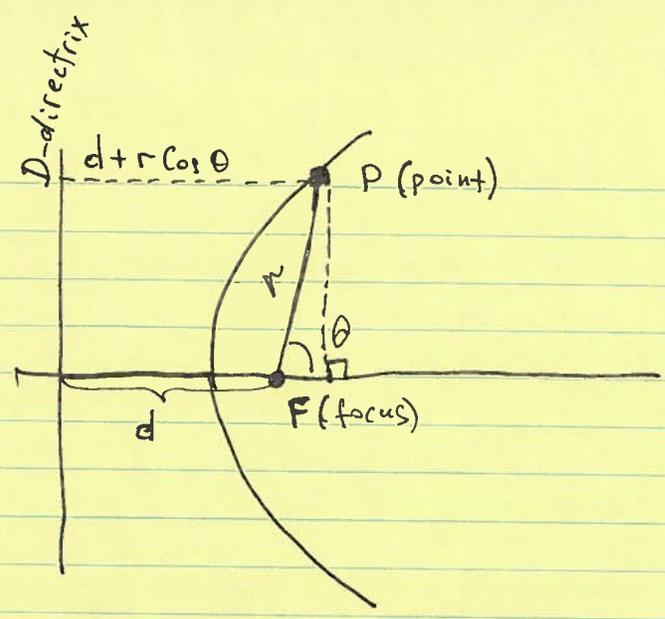
$$\varepsilon = \sqrt{1 + \frac{E l^2}{m k^2}} \Rightarrow \boxed{\frac{1}{r(\theta)} = \frac{m k}{l^2} \left( 1 - \varepsilon \cos(\theta - \theta_0) \right)}$$

can be obtained upon substituting back into energy conservation relation.

$$\frac{l^2}{m k} = \alpha - \text{constant}$$

The orbits in the Kepler problem are conical sections = hyperbola, parabola, ellipse, circle.

A definition of conical section: given a point (Focus) and a line (Directrix), each point on a conical section is such that its distance from the focus is proportional to its distance from the directrix.  $\varepsilon$  is the constant of proportionality.



If  $d$  is the distance between focus and directrix  $\Rightarrow$  from the definition above, the conical section is defined by

$$r = \epsilon \underbrace{(d + r \cos \theta)}_{PD} \Rightarrow \boxed{\frac{(\epsilon d)}{r} = 1 - \epsilon \cos \theta} \quad (**)$$

or

$$\frac{d}{r} = 1 - \epsilon \cos \theta$$

Examples:

1. Circle ( $\epsilon=0$ ): we take the limit  $\epsilon \rightarrow 0$  while sending  $d \rightarrow \infty$  so that  $\epsilon \cdot d \equiv R = \text{constant}$ . We thus get from

(\*\*):  $\boxed{\frac{R}{r} = 1} \Rightarrow R$  is the radius of the circle.  $\checkmark$

2. parabola ( $\epsilon=1$ ): Set  $\epsilon=1 \Rightarrow (**)$  yields

$$d = r - r \cos \theta = r - x \Rightarrow (d+x)^2 = r^2 = x^2 + y^2 \Rightarrow$$

$$\Rightarrow \boxed{x = \frac{1}{2d} (y^2 - d^2)} \text{ - parabola } \checkmark$$

3. Ellipse / hyperbola ( $\epsilon \neq 0, 1$ ):

We have that  $\epsilon d = r - r\epsilon \cos\theta = r - \epsilon x \Rightarrow$

$$\Rightarrow \epsilon(d+x) = r$$

$$\Rightarrow \epsilon^2(d+x)^2 = r^2 = x^2 + y^2$$

This equation is algebraically equivalent to

$$\left(x - \frac{\epsilon^2 d}{1 - \epsilon^2}\right)^2 + \frac{y^2}{1 - \epsilon^2} = \frac{\epsilon^2 d^2}{(1 - \epsilon^2)^2}$$

This equation suggests that we have

a) if  $0 < \epsilon < 1 \Rightarrow$  ellipse (closed)

b) if  $\epsilon > 1 \Rightarrow$  hyperbola (open)

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