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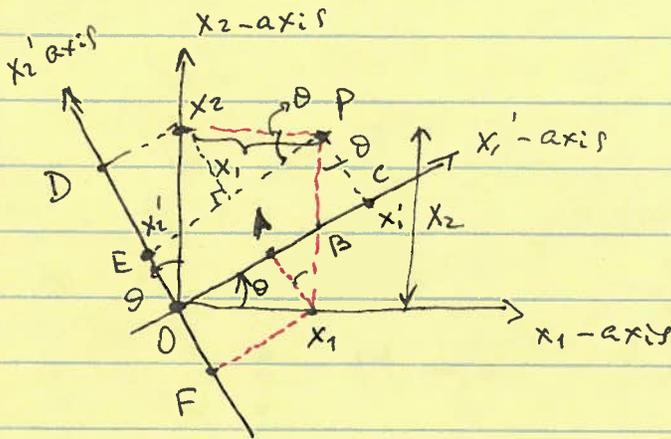
## Outline of Lecture 2:

Reminder: coordinate transform in 2D, orthogonality.

- \* scalar and vector products of 2 vectors.
- \* Differentiation of a vector wrt a scalar:  
velocity and acceleration
- \* Gradient operator
- \* Integration of vectors

Reminder:

coordinate transformations



$$x_1' = \underbrace{x_1 \cos \theta}_{OA} + \underbrace{x_2 \sin \theta}_{BC} = x_1 \cos \theta + x_2 \cos \left( \frac{\pi}{2} - \theta \right)$$

$$x_1' = OA + (AB + BC)$$

$$x_2' = \underbrace{-x_1 \sin \theta}_{DE} + \underbrace{x_2 \cos \theta}_{OD} = x_1 \cos \left( \frac{\pi}{2} + \theta \right) + x_2 \cos \theta$$

$$x_2' = OD - DE$$

$$\lambda_{11} = \cos(x_1', x_1) = \cos \theta$$

$$\lambda_{12} = \cos(x_1', x_2) = \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta$$

$$\lambda_{21} = \cos(x_2', x_1) = \cos \left( \frac{\pi}{2} + \theta \right) = -\sin \theta$$

$$\lambda_{22} = \cos(x_2', x_2) = \cos \theta$$

$$\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}$$

↓

Orthogonal matrix

$$\Lambda \cdot \Lambda^T = \mathbb{1}$$

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2$$

$$x_2' = \lambda_{21} x_1 + \lambda_{22} x_2$$

$$\left. \begin{matrix} x_1' \\ x_2' \end{matrix} \right\} \boxed{ x_i' = \sum_{j=1}^2 \lambda_{ij} x_j }, \quad i=1,2$$

## Scalar & vector products of 2 vectors:

Define: (\*) scalar product of  $\vec{A} = (A_1, A_2, A_3)$  and  $\vec{B} = (B_1, B_2, B_3)$

is  $\vec{A} \cdot \vec{B} = \sum_{i=1}^3 A_i B_i$  — sometimes called dot product.

(\*) The magnitude of  $\vec{A}$

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2} \equiv A$$

$$\frac{\vec{A} \cdot \vec{B}}{A \cdot B} = \sum_{i=1}^3 \frac{A_i}{A} \cdot \frac{B_i}{B}$$

$$\left. \begin{aligned} \frac{A_i}{A} &= \cos(\vec{A}, X_i) \equiv \lambda_i^A \\ \frac{B_i}{B} &= \cos(\vec{B}, X_i) \equiv \lambda_i^B \end{aligned} \right\} \begin{array}{l} \text{direction} \\ \text{cosines of} \\ \vec{A} \text{ and } \vec{B}. \end{array}$$

$$\begin{aligned} \frac{\vec{A} \cdot \vec{B}}{A \cdot B} &= \sum_{i=1}^3 A_i \lambda_i^A \lambda_i^B = \cos(\vec{A}, X_1) \cos(\vec{B}, X_1) + \\ &+ \cos(\vec{A}, X_2) \cos(\vec{B}, X_2) + \cos(\vec{A}, X_3) \cos(\vec{B}, X_3) \equiv \\ &\equiv \cos(\vec{A}, \vec{B}) \end{aligned}$$

So we have that  $\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos(\vec{A}, \vec{B})$ .

Problem: show that  $\vec{A} \cdot \vec{B}$  is indeed a scalar!

## \* Unit vectors:

Sometimes it is useful to describe a vector in terms of the components along the three coordinate axes. For this purpose we introduce unit vectors

\* unit vector along radial direction  $\vec{R}$  is

$$\vec{e}_R = \frac{\vec{R}}{|\vec{R}|} \quad \text{Generally } \forall \text{ vector can}$$

be represented as  $\vec{A} = A_1 \vec{e}_1 + A_2 \vec{e}_2 + A_3 \vec{e}_3 =$

$$= \sum_{i=1}^3 \vec{e}_i A_i$$

$$\Rightarrow A_i = \vec{e}_i \cdot \vec{A}$$

Since  $\vec{A} \cdot \vec{B} = AB \cos(\vec{A}, \vec{B})$ , and if any two unit vectors are orthogonal, we have

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

## Vector Product

How can we get a vector by multiplying two vectors with each other???

$$\vec{C} = \vec{A} \times \vec{B} \quad \Rightarrow \quad C_i = \sum_{j,k=1}^3 \epsilon_{ijk} A_j B_k,$$

where

$\boxed{\epsilon_{ijk}}$  is permutation symbol (or Levi-Civita symbol). It has the following properties:

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if two of these indices are equal} \\ +1 & \text{if } ijk \text{ form even perm. of } 1, 2, 3 \\ -1 & \text{if } ijk \text{ form odd perm. of } 1, 2, 3 \end{cases}$$

$$\epsilon_{122} = \epsilon_{313} = \epsilon_{211} = \dots = 0.$$

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1.$$

$$\Downarrow$$

$$C_1 = \sum_{j \neq k=1}^3 \epsilon_{ijk} A_j B_k = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2$$

$$C_2 = A_3 B_1 - A_1 B_3$$

$$C_3 = A_1 B_2 - A_2 B_1$$

Show one can show that

$$C = |\vec{C}| = \sqrt{C_1^2 + C_2^2 + C_3^2} = A \cdot B \sin \theta$$

$$\vec{C} = \vec{A} \times \vec{B}.$$

## Differentiation of a vector:

a) take a scalar,  $\phi = \phi(s)$ , and make a coordinate transformation.  $\{x_i\} \rightarrow \{x_i'\}$ .

$$\Rightarrow \phi = \phi', \quad s = s' \quad \text{so} \quad d\phi = d\phi' \quad \text{and} \\ ds = ds' \Rightarrow \frac{d\phi}{ds} = \frac{d\phi'}{ds'} = \left(\frac{d\phi}{ds}\right)' \quad \text{is also a scalar.}$$

b) similarly, components of  $\vec{A} = (A_1, A_2, A_3)$  transform as  $A_i' = \sum_{j=1}^3 \lambda_{ij} A_j$

$$\Rightarrow \frac{dA_i'}{ds'} = \frac{d}{ds'} \sum_j \lambda_{ij} A_j = \sum_j \lambda_{ij} \frac{dA_j}{ds'} = \\ = \sum_j \lambda_{ij} \frac{dA_j}{ds}$$

because  $s = s' \Rightarrow$

$\frac{dA_j}{ds}$  transform as components of a vector

$$\Rightarrow \frac{d\vec{A}}{ds} = \left( \frac{dA_1}{ds}, \frac{dA_2}{ds}, \frac{dA_3}{ds} \right) \quad \text{is a vector!}$$

Examples:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$$

$\vec{r}$  specifies the position of the particle at time  $\underline{t}$ .

In rectangular coordinates:

$$\begin{array}{l} \vec{r} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 = \sum_{i=1}^3 x_i \vec{e}_i \\ \vec{v} = \dot{\vec{r}} = \sum_i \dot{x}_i \vec{e}_i = \sum_i \frac{dx_i}{dt} \vec{e}_i \\ \vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \sum_i \ddot{x}_i \vec{e}_i = \sum_i \frac{d^2 x_i}{dt^2} \vec{e}_i \end{array} \left| \begin{array}{l} \text{position} \\ \text{velocity} \\ \text{acceleration} \end{array} \right.$$

Gradient operator:

\*  $\vec{A} \cdot \vec{B}$  - scalar

$\vec{A} \times \vec{B}$  - vector

$\frac{d\phi}{dt}$  - scalar

$\frac{d\vec{A}}{dt}$  - vector

}  $\frac{d}{dt}$  is a scalar derivative.

Is there a ~~derivative~~ vector differential operator??

YES!  $\rightarrow$  gradient operator.

Consider a scalar  $\phi(x_1, x_2, x_3) = \phi'(x_1', x_2', x_3')$ .

By chain rule of differentiation, we can write:

$$\frac{\partial \phi'}{\partial x_i'} = \sum_{j=1}^3 \frac{\partial \phi}{\partial x_j} \cdot \frac{\partial x_j}{\partial x_i'}$$

need to calculate!  
to see how his derivative transforms!

Inverse coordinate transform:  $x_j = \sum_{k=1}^3 \lambda_{kj} x'_k$ ,  $j=1, 2, 3$

Differentiating this we get:

$$\frac{\partial x_j}{\partial x'_i} = \frac{\partial}{\partial x'_i} \left( \sum_k \lambda_{kj} x'_k \right) = \sum_k \lambda_{kj} \underbrace{\left( \frac{\partial x'_k}{\partial x'_i} \right)}_{\delta_{ik}}$$

$$\Rightarrow \frac{\partial x_j}{\partial x'_i} = \sum_{k=1}^3 \lambda_{kj} \cdot \delta_{ik} \equiv \lambda_{ij}$$

So we obtained that

$$\frac{\partial \phi'}{\partial x'_i} = \sum_{j=1}^3 \lambda_{ij} \frac{\partial \phi}{\partial x_j}, \quad i=1, 2, 3$$

Bingo!  $\left( \frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3} \right)$  transforms as a vector!

Denote:  $(\text{grad})_i \equiv \nabla_i \equiv \frac{\partial}{\partial x_i} \rightarrow \text{Gradient}$

$$\vec{\text{grad}} = \vec{\nabla} = \sum_{i=1}^3 \vec{e}_i \frac{\partial}{\partial x_i}$$

Integration of vectors:

- volume
- surface
- line

The vector resulting from the volume integration of a vector function  $\vec{A} = \vec{A}(x_1, x_2, x_3)$  is

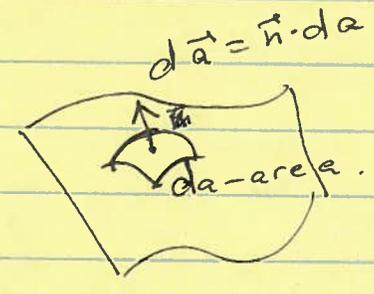
$$(*) \int_V \vec{A} dv = \left( \int_V A_1 dv, \int_V A_2 dv, \int_V A_3 dv \right) \Rightarrow$$

to integrate it we perform 3 separate integrations. In rectangular coordinates  $dv = dx_1 dx_2 dx_3$ .

Surface integration:

(\*) The integral over surface S of the projection of  $\vec{A}$  onto the normal to that surface is

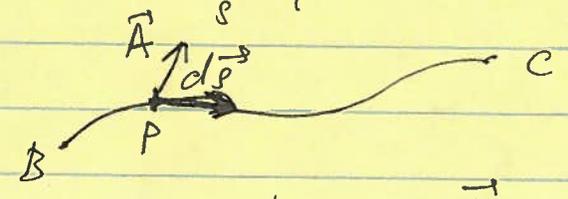
$$\int_S \vec{A} \cdot d\vec{a}, \text{ where } d\vec{a} = \vec{n} da$$



$$\Rightarrow \left. \begin{aligned} da_1 &= dx_2 dx_3 \\ da_2 &= dx_1 dx_3 \\ da_3 &= dx_1 dx_2 \end{aligned} \right\}$$

$$\int_S \vec{A} \cdot d\vec{a} = \int_S \vec{A} \cdot \vec{n} da = \int_S \sum_i A_i d\vec{a}_i$$

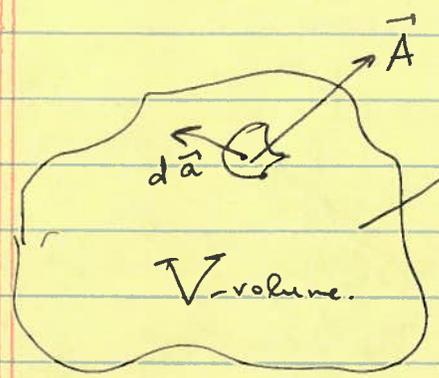
(\*) Line integral.



$$\int_{B-C} \vec{A} d\vec{s} = \int_{B-C} \sum_i A_i dx_i, \text{ where } d\vec{s} = (dx_1, dx_2, dx_3)$$

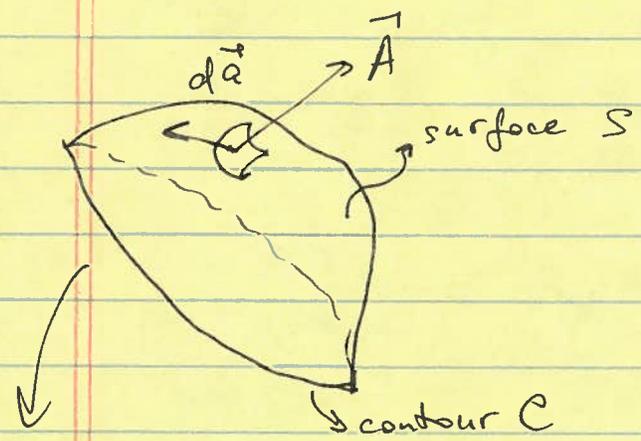
# Relations between integrals:

1. Gauss's theorem relates surface integrals to volume integrals:



$$\int_S \vec{A} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{A} \cdot dV$$

2. Stokes's theorem relates surface integrals to line integrals:



$$\int_C \vec{A} \cdot d\vec{s} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

$\underbrace{\hspace{10em}}$ 
 $\underbrace{\hspace{10em}}$

line Int
surface int.

open surface S  
with contour C.