

Lecture 8

Outline:

- * Principles of superposition: Fourier series
- * Conservative quantities and conservative systems.

Principles of superposition -
- Fourier series

The Forced oscillations we have been discussing
obey the diff. equation

$$\left(\frac{d^2}{dt^2} + \alpha \frac{d}{dt} + \kappa \right) X(t) = A \cos \omega t$$

Linear operator \hat{L} .

Define a linear operator, $\hat{L} = \frac{d^2}{dt^2} + \alpha \frac{d}{dt} + \kappa$.
It satisfies

$$\begin{cases} \hat{L}[f(t) + g(t)] = \hat{L}f(t) + \hat{L}g(t) & \text{principle of superposition} \\ \hat{L}(c \cdot f(t)) = c \cdot \hat{L}f(t), \quad c = \text{const.} \end{cases}$$

Therefore, if we have two solutions x_1, x_2

such that $\hat{L}x_1 = F_1(t)$, $\hat{L}x_2 = F_2(t)$, then

$$\hat{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F_1(t) + \alpha_2 F_2(t).$$

We can extend this to a set of solutions $x_n(t)$:

$$\hat{L}\left(\sum_{n=1}^{\infty} \alpha_n x_n(t)\right) = \sum_{n=1}^{\infty} \alpha_n F_n(t)$$

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If we identify our equation as

$$x(t) = \sum_{n=1}^{\infty} \alpha_n x_n(t)$$

$$F(t) = \sum_{n=1}^{\infty} \alpha_n F_n(t),$$

then $L x(t) = F(t).$

If each $F_n(t)$ has a simple harmonic dependence on t : $\omega_n t$, then

$$F(t) = \sum_n \alpha_n \cos(\omega_n t - \phi_n),$$

The steady state solution is

$$x(t) = \frac{1}{m} \sum_n \frac{\alpha_n}{\sqrt{(\omega_0^2 - \omega_n^2)^2 + 4\omega_n^2 \beta^2}} C_n (\omega_n t - \phi_n - \delta_n)$$

with $\delta_n = \tan\left(\frac{2\omega_n \beta}{\omega_0^2 - \omega_n^2}\right)$.

If $f \Rightarrow$ periodic $F(t+\tau) = F(t)$, $\tau = \frac{2\pi}{\omega}$,

then $F(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t)$

Fourier Theorem.

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where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t') \cos nt' dt'$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t') \sin nt' dt'$$

Part 2:

Reminder: Consider two isolated bodies 1 and 2

According to Newton's III Law:

$$\vec{F}_1 = -\vec{F}_2 \Rightarrow m_1 a_1 = -m_2 a_2$$

$$m_1 \frac{d\vec{v}_1}{dt} = -m_2 \frac{d\vec{v}_2}{dt}$$

$$\frac{d(m_1 \vec{v}_1 + m_2 \vec{v}_2)}{dt} = 0$$

$$\Rightarrow \vec{P}_1 + \vec{P}_2 = \text{constant vector}$$

Conservation of momentum.

Conservative systems: 1-Dimensions (1D)

In general (arbitrary dimension) a force $\vec{F}(\vec{r})$ is called conservative if there is a function $V(\vec{r})$ such that

$$(*) \quad \vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r}) = -\overrightarrow{\text{grad}} V(\vec{r}).$$

Then $V(\vec{r})$ is called potential. In components one has $F_{x_1} = -\frac{\partial V}{\partial x_1}$, $F_{x_2} = -\frac{\partial V}{\partial x_2}$, $F_{x_3} = -\frac{\partial V}{\partial x_3}$, ...

In 1D, all forces that depend on the coordinate x only, are conservative. This can be seen by solving Eq (*) for the potential

ex 2a

$$V(x) = - \int_{x_0}^x F(x') dx' -$$

-which is a definite integral. Note that by changing the lower integration limit we add a constant to $V(x)$.

- Such a constant is physically irrelevant
 - The potential is defined up to an additive constant.
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Suppose we have a 1D system with force $F(x)$ and potential $V(x)$.

in 1D

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The equation of motion reads:

$$\frac{m \frac{dv}{dt}}{dt} = F(x).$$

We can make use of the trick we applied in solving the problem of the simple Harmonic Oscillator:

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = \frac{1}{2} \frac{d}{dx} (v^2)$$

using $F(x) = -\frac{dV(x)}{dx}$ (v is the velocity
 V is the potential)

We have that our equation of motion is equivalent to

$$\frac{d}{dx} \left(\frac{1}{2} m v^2 \right) = -\frac{dV(x)}{dx} \Rightarrow \frac{d}{dx} \left(\frac{1}{2} m v^2 + V(x) \right) = 0.$$

We call $\frac{1}{2} m v^2 \rightarrow$ Kinetic energy.

Therefore we obtain that

$$\frac{1}{2} m v^2 + V(x) = E = \text{constant}$$

Kinetic energy Potential energy Total energy of the system

E is an integration constant, so it depends on the initial ~~other~~ conditions.

One can proceed and solve the equation of motion for $X(t)$ by integrating:

$$\frac{1}{2} m \left(\frac{dx}{dt} \right)^2 = E - V(x) \quad \text{and we separate}$$

$$\int_{x(t_0)}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m} [E - V(x')]}} = \int_{t_0}^t dt' = t - t_0$$

Example: The Harmonic oscillator

Let us use the method described above (energy conservation!) to find $X(t)$ for a harmonic oscillator.

For simplicity assume that at $t=0$, $X(t=0)=0$, and $\dot{X}(t=0)=v_0$.

As the first step, we find the potential. We have

$$F(x) = -kx \quad \text{and} \quad V(x) = - \int_0^x F(x') dx'$$

Integration yields $V(x) = \frac{1}{2} kx^2$ (+ a constant, that we set to zero).

Therefore $x(t)$ is found from

$$x(t) = \sqrt{\frac{m}{2}} \int_{x(t_0)}^t \frac{dx'}{\sqrt{E - \frac{k(x')^2}{2}}} = t - t_0.$$

Next we factor $\frac{1}{\sqrt{E}}$ out of the integrand

on the left-hand side and define a new variable (y) such that

$$y^2 = \frac{k}{2E} \cdot (x')^2 \Rightarrow y = \sqrt{\frac{k}{2E}} x', \quad dx' = \sqrt{\frac{2E}{k}} dy$$

The equation giving $x(t)$ becomes

$$\sqrt{\frac{m}{2}} \cdot \frac{1}{\sqrt{E}} \sqrt{\frac{2E}{k}} \int_0^t \frac{\sqrt{\frac{k}{2E}} \cdot x(t)}{\sqrt{1-y^2}} dy = t$$

One more change of variables: $y = \sin \theta$, $dy = \cos \theta d\theta$
gives the final result

$$t = \sqrt{\frac{m}{k}} \int_0^{\arcsin(\sqrt{\frac{k}{2E}} x(t))} \frac{\cos \theta d\theta}{\sqrt{1-\sin^2 \theta}} = \sqrt{\frac{m}{k}} \arcsin \left[\sqrt{\frac{k}{2E}} x(t) \right]$$

So that we find the solution

$$x(t) = \sqrt{\frac{2E}{k}} \sin \left(\sqrt{\frac{k}{m}} t \right).$$

Finally we remember that

$$\sqrt{\frac{k}{m}} = \omega \text{ and that } \frac{mv^2}{2} + V(x) = E$$

and that at $x=0, t=0, v=v_0$.

$$\text{Since } V(x=0) = 0 \Rightarrow E = \frac{mv_0^2}{2}$$

We thus obtain

$$x(t) = \sqrt{\frac{2}{k} \cdot \frac{mv_0^2}{2}} \sin(\omega t) = \frac{v_0}{\omega} \sin(\omega t)$$

That is the solution we were looking for.

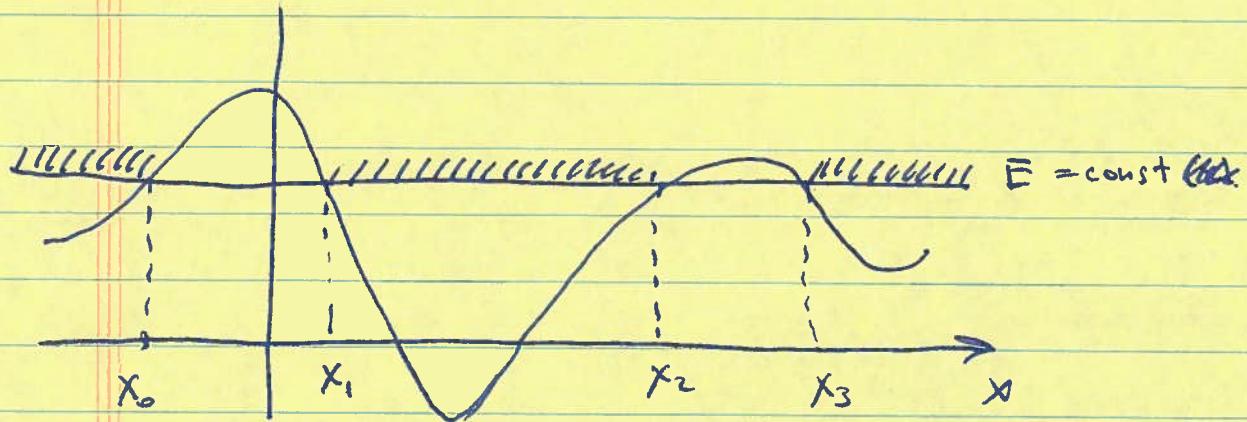
So the solution of the equations of motion is determined if we are able to compute the primitive of

$$\frac{1}{\sqrt{\frac{2}{m}(E-V(x))}}, \text{ i.e.}$$

$$\int_{x(t_0)}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}(E-V(x))}} = t - t_0.$$

But even if we cannot compute the primitive we can study qualitatively the motion.

- Let us graph $V(x)$ vs x and E :



Since $\frac{1}{2}mv^2 > 0$, we know that motion can happen only for $E > V(x)$. In our example this is for $x < x_0$, $x_1 < x < x_2$, and $x > x_3$.

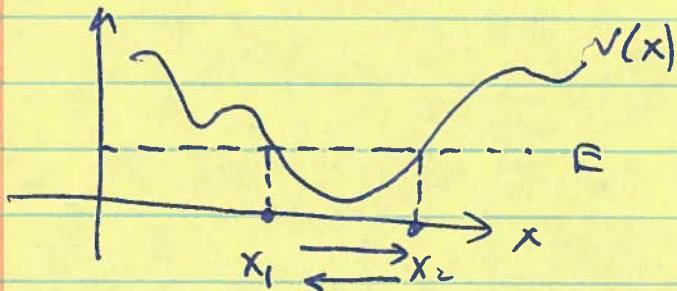
Points where $V(x) = E$ are called turning points, because $v=0$ there, i.e. v changes the sign!

Note in particular that motion between two turning points is periodic.

The period is computed as

$$T = 2 \int_{x_1}^{x_2} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x))}}$$

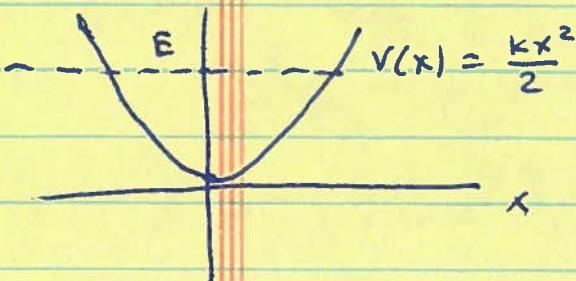
factor 2 comes from the fact that one period corresponds to two trips between the turning points x_1 and x_2 .



Note then in general T depends on E .

Example of calculation of a period:

The Harmonic oscillator $\Rightarrow V(x) = \frac{kx^2}{2}$ - the potential energy.



$$T = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E - \frac{k}{2}x^2)}} =$$

$$= 2 \sqrt{\frac{m}{2}} \cdot \frac{1}{\sqrt{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \frac{kx^2}{2E}}} ,$$

where x_1 and x_2 are determined by

$$E = \frac{kx^2}{2} \Rightarrow x_{1,2} = \pm \sqrt{\frac{2E}{k}} \text{ - turning points.}$$

We define a new integration variable y : $\frac{kx^2}{2E} = y^2$

$$\Rightarrow x = y \sqrt{\frac{2E}{k}} \Rightarrow y_{1,2} = \pm 1$$

$$dx = \sqrt{\frac{2E}{k}} \cdot dy$$

and $T = 2 \sqrt{\frac{m}{2E}} \int_{-1}^1 \frac{dy \sqrt{2E/k}}{\sqrt{1-y^2}} = 2 \sqrt{\frac{m}{k}} \int_{-1}^1 \frac{dy}{\sqrt{1-y^2}}$

Finally the integral is computed by setting $y = \cos \theta$

and we obtain $T = 2 \sqrt{\frac{m}{k}} \pi = \frac{2\pi}{\omega}$.