

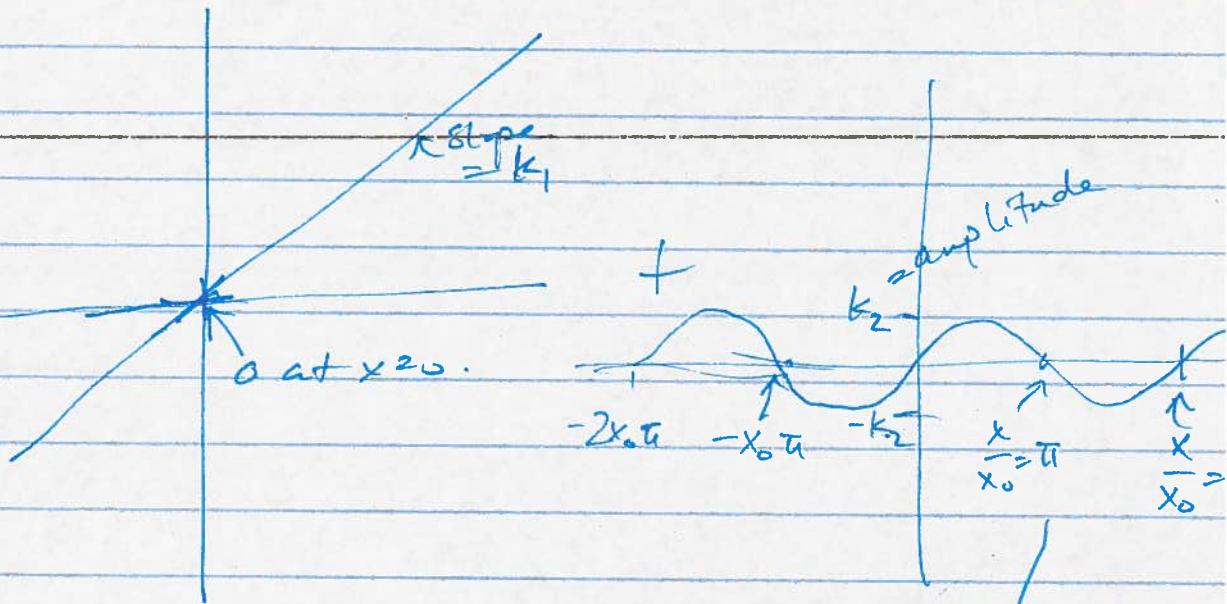
PHYS 421 MIDTERM EXAM 1 - Solutions

$$1. V(x) = \underbrace{k_1 x}_{\text{linear in } x} + k_2 \sin\left(\frac{x}{x_0}\right)$$

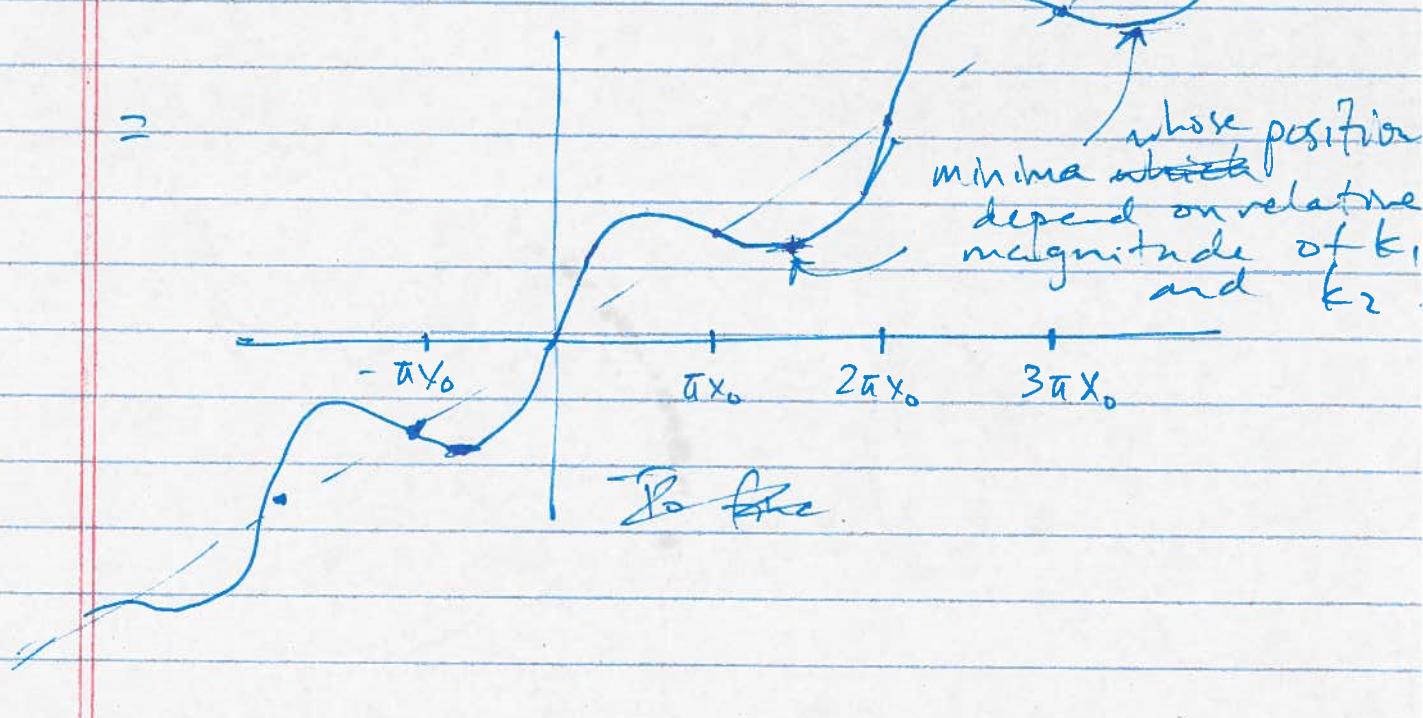
linear
in x .

oscillating in x & spatial
frequency $k = 1/x_0$.
zero at $x=0$.

(a)



=



(b) Potential $V(x)$ has dimensions of energy,
or force * distance.

$[x] = \text{distance}$.

$\left[\sin\left(\frac{x}{x_0}\right) \right]$ is dimensionless.

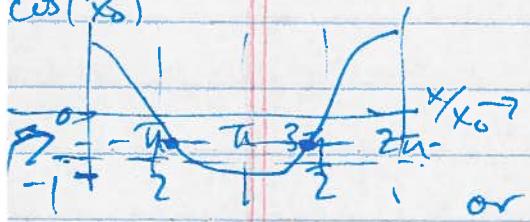
$\Rightarrow k_1$ has 1 dimension of force.

k_2 has dimension of energy,
or force times
distance.

(c) The system will admit equilibrium
points if there are extrema in $V'(x)$:

$$V'(x) = k_1 + k_2 \cos\left(\frac{x}{x_0}\right) = 0 \text{ at :}$$

$$\cos\left(\frac{x}{x_0}\right)$$



$$\cos\left(\frac{x}{x_0}\right) = -\frac{k_1 x_0}{k_2} \quad \left. \begin{array}{l} \text{if } k_1, k_2, x_0 \\ \text{all positive} \end{array} \right\}$$

$$\frac{x}{x_0} = \arccos\left(-\frac{k_1 x_0}{k_2}\right)$$

$$\left| \frac{k_1 x_0}{k_2} \right| = \frac{k_1 x_0}{k_2}$$

must fall
between 0 and 1.

$$x_0 \text{ and } \arccos\left(-\frac{k_1 x_0}{k_2}\right)$$

values of the
parameters
such that

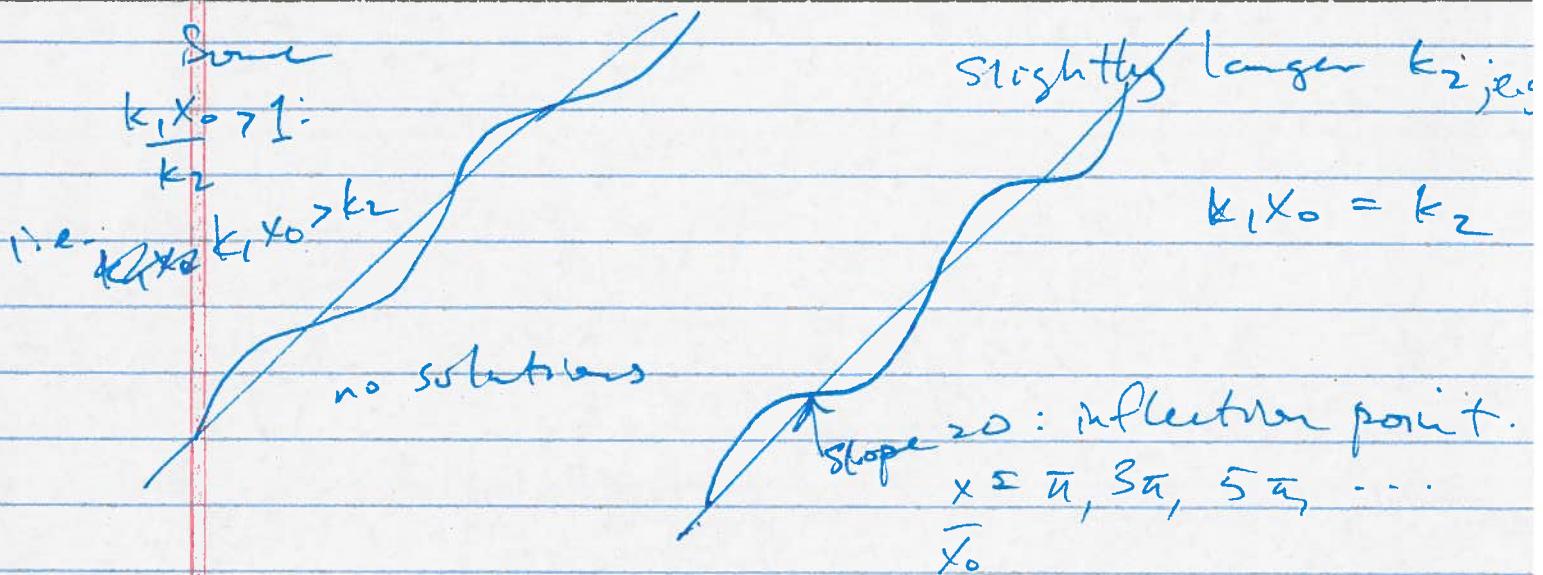
$\cos\left(\frac{x}{x_0}\right)$ is a negative
number and

falls between 0 and
-1:

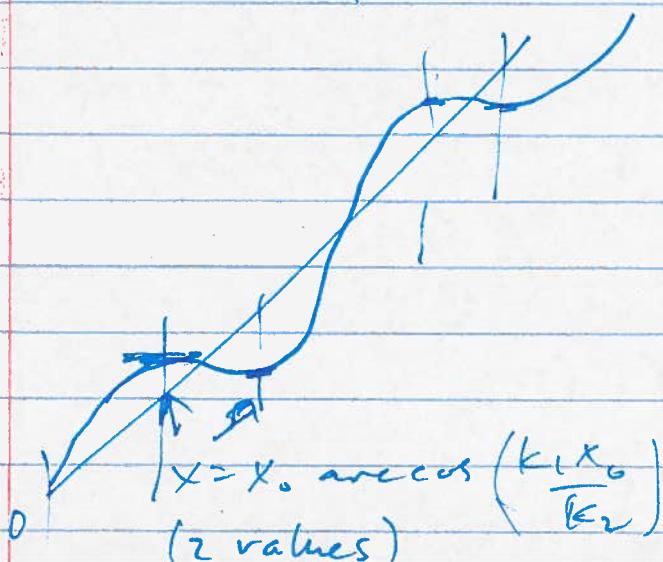
$$0 < \frac{k_1 x_0}{k_2} < 1.$$

For all values of $\frac{k_1 x_0}{k_2}$ that

satisfy this there will be two extrema,
 (one the \min and one the local
 max, for each cycle of 2π); except
 for $\frac{k_1 x_0}{k_2} = 1$, which occurs
 corresponds to the
 min and max coinciding,
 i.e. an inflection point,



$k_1 x_0 < k_2$:



(*) The stable equilibrium point that is nearest to the origin will occur for x between 0 and $-\bar{a}$, since between x_0

0 and $+\bar{a}$ only the unstable equilibrium will occur; if the system admits equilibria at all.

$$x_0 = \text{the solution for } \frac{x}{x_0} = \cos\left(\frac{k_1 x_0}{k_2}\right)$$

that falls between 0 and $\frac{\pi}{2}$ minus \bar{a} :

$$x_{\min, \text{equl}} \text{ closest to origin} = x_0 \left[\cos\left(\frac{k_1 x_0}{k_2}\right) - \bar{a} \right]$$

We have that the frequency of small oscillations about a stable equilibrium, i.e. if E is just a small very slightly larger than $V(x_0)$ where x_0 is the equilibrium point, is (obtained in class by Taylor expansion):

$$\omega^2 = \underline{V''(x_0)}$$

i.e. ignore higher order terms $(x-x_0)^3, \dots$
 Here, $V''(x) = \frac{d}{dx} V'(x) = \left(\frac{k_2}{x_0}\right) \left(-\sin\left(\frac{x}{x_0}\right)\right) \left(\frac{1}{x_0}\right)$
 $= -\frac{k_2}{x_0^2} \sin\frac{x}{x_0}$.

At our stable equilibrium,

$$x_e = x_0 \left(\arccos\left(\frac{k_1 x_0}{k_2}\right) - \bar{u} \right)$$

$$V''(x_e) = -\frac{k_2}{x_0^2} \sin\left(\arccos\left(\frac{k_1 x_0}{k_2}\right) - \bar{u}\right)$$

$$= + \frac{k_2}{x_0^2} \sin\left(\arccos\left(\frac{k_1 x_0}{k_2}\right)\right)$$

$$\frac{\cos^{-1}\left(\frac{k_1 x_0}{k_2}\right)}{\sqrt{k_2^2 - k_1^2 x_0^2}} \cdot \frac{1}{k_2} = \sqrt{1 - \left(\frac{k_1 x_0}{k_2}\right)^2}$$

and $\omega^2 = \frac{V''(x_e)}{m}$ $\left| \omega = \sqrt{\frac{k_2}{x_0^2 m} \sqrt{1 - \left(\frac{k_1 x_0}{k_2}\right)^2}} \right|$.

$$\text{and } f = \omega / 2\pi.$$

2. We see that this problem is much like the example of drag force that we did in 2nd week of lectures, where the force always opposes the motion whatever the direction of the motion and in a way depending on the magnitude of the motion ($\propto -v$), but here rather than linear in velocity magnitude, the force scales with some power other than 1. i.e. $|v|^\alpha$. = 1-D problem

1 Equation:

$$(a) m\ddot{x} = -\gamma |x|^\alpha \quad \text{lets take positive } x \text{ direction, then can get rid of } |\cdot| \text{ i.e. } x \geq 0.$$

$$\Rightarrow m\ddot{x} = -\gamma x^\alpha$$

$$\ddot{x} = -\frac{\gamma}{m} x^\alpha$$

$$\text{let } \frac{dv}{dt} = -\frac{\gamma}{m} v^\alpha$$

inspired by

$$\text{know: } \alpha > 0, \text{ constant} \\ \gamma > 0, \text{ constant} \\ v(t=0) = v_0 > 0.$$

Separation of variables, as we did in class for $\alpha = 1$

$$\int \frac{dv}{v^\alpha} = -\frac{\gamma}{m} \int dt \quad \text{let } -\frac{\gamma}{m} t =$$

$$v(t=0)$$

$$\left[\frac{1}{1-\alpha} v^{1-\alpha} \right]_{v_0}^{v(t)} = -\frac{\gamma}{m} t, \quad \cancel{v_0}$$

$$\Rightarrow (v(t))^{1-\alpha} - v_0^{1-\alpha} = (\alpha-1) \frac{\gamma}{m} t$$

$$\text{and } v(t)^{1-\alpha} = (\alpha-1) \frac{\gamma}{m} t + v_0^{1-\alpha}$$

$$\therefore v(t) = \left((\alpha-1) \frac{\gamma}{m} t + v_0^{1-\alpha} \right)^{\frac{1}{1-\alpha}}.$$

(b) Qualitative plot of $v(t)$ for $\alpha < 1$:

At $t=0$, $v(t=0) = (v_0^{1-\alpha})^{\frac{1}{1-\alpha}} = v_0$, as expected.

For $t > 0$, since $\alpha < 1$, $1-\alpha > 0$, $\alpha-1 < 0$.

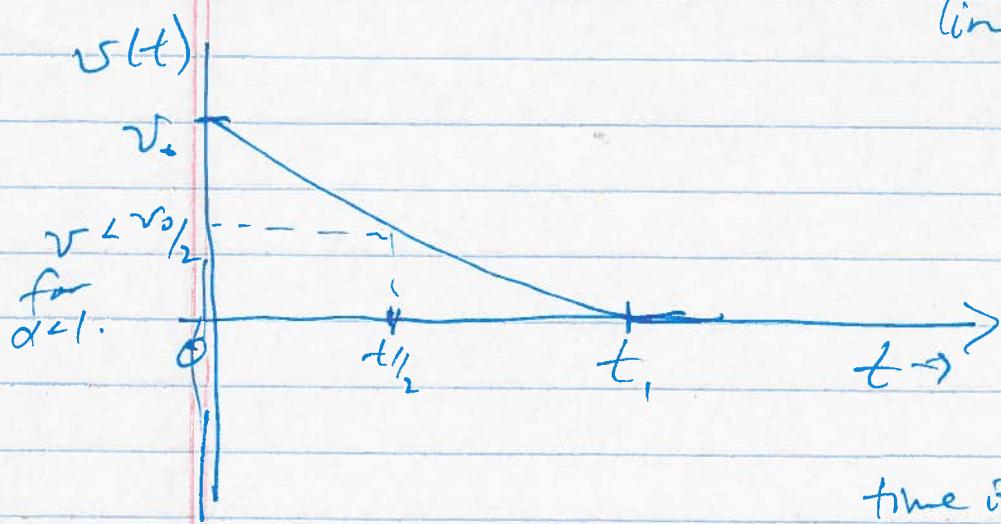
$\therefore (\alpha-1) \frac{\gamma}{m} t < 0$, since γ is a positive constant.

Thus, the term in () decreases with time.

Moreover, there will be some time at which it goes to zero, for a given α :

i.e. $\left(v_0^{1-\alpha} - \underbrace{(1-\alpha) \frac{\gamma}{m} t}_{\geq 0 \geq 0} \right)$ decreases linearly in the

($\sqrt[1-\alpha]{\cdot}$ decreases faster than linearly in time. (can show this)).



$$\text{at } t_1, (\alpha-1) \frac{\gamma}{m} t_1 = v_0^{1-\alpha}$$

$$t_1 = \frac{m}{\gamma} \frac{v_0^{1-\alpha}}{1-\alpha}$$

time it takes for the particle to stop.

(c) For $\alpha > 1$, $\alpha - 1 > 0$, and the $(\alpha - 1)\frac{\delta}{m}t$ term increases linearly with time. Meanwhile, $1 - \alpha < 0$, $\therefore \sqrt[1-\alpha]{v_0} = \frac{1}{v_0^{\alpha-1}}$

and the power on the (\quad) behaves as

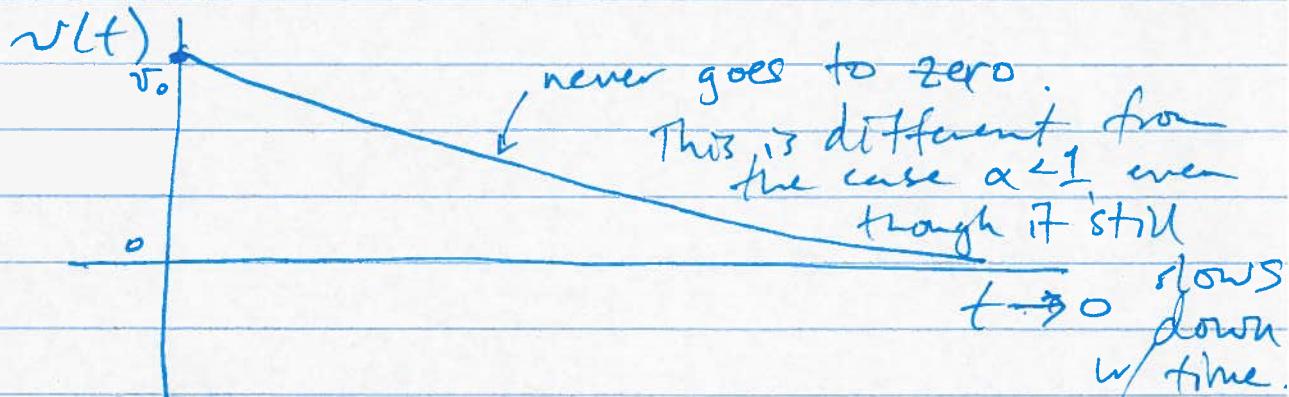
$$\underbrace{\frac{1}{1-\alpha}}_{<0} = -\underbrace{(\alpha-1)}_{>0} = \frac{1}{(\quad)^{1/\alpha-1}}$$

i.e. $v(t) = \frac{1}{(\underbrace{(\alpha-1)\frac{\delta}{m}t - \frac{1}{v_0^{\alpha-1}})^{1/\alpha-1}}}$

increases
linearly
w/ time constant
in time

Thus, as time increases, the denominator increases, enhanced by the power $1/\alpha-1$ which is > 1 for $1 < \alpha < 2$.

\therefore Sketch:



$$\lim_{t \rightarrow \infty} v = 0$$

(d) For $\alpha < 1$, the particle travels for time $t_1 = \frac{m}{\gamma} v_0^{\frac{1-\alpha}{1-\alpha}}$, at which point it stops.

The distance travelled is $x(t_1)$

$$\int_{x(0)}^{x(t_1)} dx = X_{\text{tot}} = \int_0^{t_1} v(t) dt = \int_0^{t_1} \left(v_0^{1-\alpha} - (1-\alpha) \frac{\gamma}{m} t \right)^{\frac{1}{1-\alpha}} dt$$

let: $\frac{\gamma}{m}(1-\alpha) = b \quad (>0)$

$$v_0^{1-\alpha} = a \cdot \underbrace{\gamma}_{\text{and } v_0} \quad \left. \begin{array}{l} t_1 = \frac{a}{b} \\ \frac{1}{1-\alpha} = c \quad \underbrace{\gamma}_{\text{and }} \\ \text{also } > 1 \end{array} \right\}$$

$$\Rightarrow X_{\text{tot}} = \int_0^{a/b} (a - bt)^c dt = \left. \frac{(a - bt)^{c+1}}{(c+1)(-b)} \right|_{t=0}^{t=a/b}$$

at $t = a/b$, $bt = a$ and $a - bt = 0$.

at $t=0$, $bt=0$ and $a - bt = a$.

since $b > 0$ for $\alpha < 1$, $-b$ less than 0. \Rightarrow switch limits

$$\therefore X_{\text{tot}} = \left. \frac{(a - bt)^{c+1}}{b(c+1)} \right|_{t=a/b}^{t=0} = \frac{a^{c+1}}{b(c+1)}$$

where $c = \frac{1}{1-\alpha}$; $c+1 = \frac{1}{1-\alpha} + 1 = \frac{1+1-\alpha}{1-\alpha} = \frac{2-\alpha}{1-\alpha}$.

and $a^{c+1} = (v_0^{1-\alpha})^{\frac{2-\alpha}{1-\alpha}} = v_0^{2-\alpha}$.

$b \cdot (c+1) = \frac{\gamma}{m}(1-\alpha) \cdot \left(\frac{2-\alpha}{1-\alpha} \right) = \frac{\gamma}{m}(2-\alpha)$.

$$\therefore x_{\text{tot}} = \frac{v_0^{2-\alpha}}{\frac{r}{m}(2-\alpha)} = \left[\frac{m}{r} \frac{v_0^{2-\alpha}}{(2-\alpha)} \right].$$

(e) For values of $\alpha > 1$, let us re-write the integral to again make explicit all values > 0 : start with $v(t)$:

$$v(t)^2 / ((\alpha-1)) \underbrace{\frac{r}{m} t + v_0^{1-\alpha}}_{>0}^{\frac{1}{1-\alpha}}$$

for $\alpha > 1$, $1-\alpha < 0$; let $1-\alpha = (-1)(1-\alpha)$:

$$\Rightarrow v(t) = \left(\underbrace{(\alpha-1)}_{>0} \underbrace{\frac{r}{m} t + v_0^{-\alpha+1}}_{>0} \right)^{-\frac{1}{\alpha-1}}$$

increases
w/ time

Now, our integral becomes (since there is no t , at which $v \rightarrow 0$ (finite time), we must integrate to $t \rightarrow \infty$):

$$x_{\text{tot}} = \int_{t=0}^{\infty} v(t) dt = \int_0^{\infty} \left((\alpha-1) \frac{r}{m} t + v_0^{-\alpha+1} \right)^{-\frac{1}{\alpha-1}} dt$$

$$\text{(let us take } a = v_0^{-\alpha+1}; b = (\alpha-1) \frac{r}{m} \text{ (>0)}$$

and $c = -\frac{1}{\alpha-1}$

$$\therefore x_{\text{tot}} = \int_{t=0}^{\infty} (at + bt)^c dt = \left. \frac{(at+bt)^{c+1}}{(c+1)b} \right|_{t=0}^{\infty} \quad \begin{array}{l} \text{for } c \neq -1, \\ \text{i.e. not} \\ \text{for } \alpha = 2 \end{array}$$

$\alpha = 2$:

$$\text{For } \alpha = 2, \alpha - 1 = 1 \text{ and } \frac{-1}{\alpha - 1} = -1$$

Integral becomes

$$\int_{t=0}^{\infty} \frac{dt}{a+bt} = \frac{1}{b} \ln(a+bt) \Big|_{t=0}^{\infty}$$

\rightarrow goes to ∞ at $\alpha = 2$.

For all other values of $\alpha > 1$, let us treat separately $1 < \alpha < 2$; here,
 $c = \frac{-1}{\alpha-1}$ ranges from -1 to $-\infty$, (at $\alpha=2, c=-1$),
 $\therefore c+1$ always < 0 for $1 < \alpha < 2$.
 $c+1 > 0$ for $\alpha > 2$.

$1 < \alpha < 2$:

$$X_{\text{tot}} = \frac{(a+bt)^{c+1}}{(c+1)b} \Big|_{t=0}^{\infty} = \frac{(a+bt)^{c+1}}{-(c+1)b} \Big|_{t=\infty}^{t=0}, \text{ where } c+1 = \frac{2-\alpha}{\alpha-1} \neq 1$$

$$= \frac{\alpha-2}{\alpha-1}$$

For since $1 < \alpha < 2$, $\alpha-2 < 0$, $\alpha-1 > 0$, can
write $-(c+1) = \frac{2-\alpha}{\alpha-1}$ always > 0 .

$$\text{Now, } -(c+1)b = \frac{(2-\alpha)}{(\alpha-1)} (\alpha-1) \frac{b}{m} = \frac{(2-\alpha)}{m}. \quad (20)$$

$$\text{At } t=0, (a+bt)^{c+1} = a^{c+1} = (v_0^{-(\alpha-1)})^{\frac{\alpha-2}{\alpha-1}} = v_0^{2-\alpha}$$

$$\text{as } t \rightarrow \infty, (a+bt)^{c+1} = (v_0^{-(\alpha-1)} + (\alpha-1) \frac{b}{m} t)^{\frac{-(2-\alpha)}{\alpha-1}} \rightarrow 0.$$

$$\therefore X_{\text{tot}} = \frac{m}{b} \frac{v_0^{2-\alpha}}{(2-\alpha)}, \text{ which is the same result}$$

that we obtained for $\alpha < 1$.

\Rightarrow finite distance, for $0 < \alpha < 2$.

For $\alpha > 2$: Now our integral does not have the limits switched, since $c+1 > 0$ for all $\alpha > 2$.

$$\therefore X_{tot} = \frac{(at+bt)^{c+1}}{(c+1)b} \Big|_{t=0}^{\infty} \rightarrow \infty \text{ as } t \rightarrow \infty.$$

\therefore For all $\alpha \geq 2$, the particle travels an infinite distance.

For all $\alpha < 2$, the particle travels a finite distance, $X = \frac{m v_0^{2-\alpha}}{\gamma(2-\alpha)}$.

Note that $\alpha=1$ corresponds to the case we did in class, for which the distance travelled, for $t \gg \frac{m}{\gamma}$, was $\left[\frac{m v_0}{\gamma} \right]$. Our result above reduces to this same result for $\alpha=1$. \therefore Result crosses smoothly through $\alpha=1$, for all $0 < \alpha < 2$.

3. (a) Let general solution be

$$x(t) = A \cos(\omega t) + B \sin(\omega t)$$

(can also arrive at same result w/ gen. soln.
 $x(t) = A \cos(\omega t + \phi)$.)

$$\dot{x}(t) = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

Initial conditions: at $t = -\frac{\pi}{\omega}$ i.e. at $\omega t = -\pi$,

$$(i) x(t) = a \text{ and } (ii) \dot{x}(t) = 0$$

From (ii), $B = 0$ since $\sin(-\pi) = 0$ and $\cos(-\pi) = -1$.

from (i), $A = -a$.

$$\Rightarrow x(t) = -a \cos(\omega t)$$

$$\ddot{x}(t) = a\omega^2 \cos(\omega t). \text{ Check: } \ddot{x}(t) + \omega^2 x(t) = 0 \checkmark$$

(b) For the next half cycle (from $\omega t = 0$ to $\omega t = \pi$)

~~driven~~ oscillator driven by $F(t) = Qt$. $Q \geq 0$ const.

Told to guess particular solution

$$x(t) = Bt^\beta$$

$$\therefore x_{\text{PART}}(t) = Bt^\beta \text{ in } \ddot{x} + \omega^2 x = \frac{Qt}{m} \therefore$$

$$\begin{aligned} \dot{x}_{\text{PART}}(t) &= B \cdot \beta t^{\beta-1} \\ \ddot{x}_{\text{PART}}(t) &= B \cdot \beta \cdot (\beta-1) t^{\beta-2} \end{aligned} \quad \left. \begin{aligned} B\beta(\beta-1)t^{\beta-2} + \omega^2 Bt^\beta &= \frac{Qt}{m} \\ \text{can only be true} \end{aligned} \right\}$$

if $\beta = 1$: $\beta-1 = 0$.

$$\Rightarrow \omega^2 Bt = \frac{Qt}{m} \Rightarrow Q = \frac{\omega^2}{m} B$$

$$\therefore x_{\text{PART}}(t) = \frac{Q}{m\omega^2} t$$

(c) For ~~and~~
 $Q t < \pi/\omega$

$$x(t) = x_{\text{hom}}(t) + x_{\text{PART}}(t) = -a \cos \omega t + \frac{Q}{m\omega^2} t$$

(c) For $0 < t < \frac{\pi}{\omega}$, general solution is:

$$x_{\text{driven}}(t) = A \cos(\omega t) + B \sin(\omega t) + \frac{Q}{m\omega^2} t.$$

i.e. Need for our oscillator to have, just after
prior motion t=0, the same position and velocity as
sets initial conditions: so using result from (a) for
the undriven oscillator,
prior motion
sets initial conditions
for driven motion

$$x(t_{0-}) = -a \text{ since } \cos(0) = 1.$$

$$\dot{x}(t_{0-}) = 0 \text{ since } \sin(0) = 0.$$

Must match these just after $t=0$ for our driven oscillator:

$$x_{\text{driven}}(t) = A (\text{since } \sin(0) = 0 = t) = -a \\ \Rightarrow A = -a.$$

$$\dot{x}_{\text{driven}}(t_{0+}) = B\omega \cos(\omega t_0) + \frac{Q}{m\omega^2} = 0$$

$$\Rightarrow B\omega = -\frac{Q}{m\omega^2}, B = -\frac{Q}{m\omega^3}.$$

∴ $x_{\text{driven}}(t)$ must be: (i.e. for $0 < t < \frac{\pi}{\omega}$).

$$x_d(t) = -a \cos(\omega t) - \frac{Q}{m\omega^3} \sin(\omega t) + \frac{Q}{m\omega^2} t.$$

(d) For $t > \frac{\pi}{\omega}$, once again the prior motion sets initial conditions for the motion for $t > \frac{\pi}{\omega}$, which will have general solution

$$x(t > \frac{\pi}{\omega}) = A \cos(\omega t) + B \sin(\omega t).$$

These conditions are set by evaluating $x_d(t)$ and $\dot{x}_d(t)$ at $t = \frac{\pi}{\omega}$ i.e. at $\omega t = \pi$ where $\sin \omega t = 0$ and $\cos \omega t = -1$:

$$x_d\left(\frac{\pi}{\omega}\right) = -a(-1) + \frac{Q\pi}{m\omega^3} = a + \frac{Q\pi}{m\omega^3}.$$

$$\dot{x}_d\left(\frac{\pi}{\omega}\right) = aw \cancel{\sin^2 \pi} - \frac{Qw \cos \pi}{m\omega^3} + \frac{Q}{m\omega^2} = \frac{2Q}{m\omega^2}.$$

∴ Our general solution for our undriven oscillator for $t > \frac{\pi}{\omega}$ can be written specifically using these initial conditions, i.e. at $t = \frac{\pi}{\omega}$, position $= a + \frac{Q\pi}{m\omega^3}$ and velocity $\dot{x} = \frac{2Q}{m\omega^2}$.

$$x\left(t = \frac{\pi}{\omega}\right) = A \underbrace{\cos \pi}_{-1} + B \underbrace{\sin \pi}_{0} = -A = a + \frac{Q\pi}{m\omega^3}$$

$$\Rightarrow A = -a - \frac{Q\pi}{m\omega^3}.$$

$$\dot{x}\left(t = \frac{\pi}{\omega}\right) = -Aw \cancel{\sin^2 \pi} + Bw \cos \pi = \frac{2Q}{m\omega^2}$$

$$\Rightarrow B = \frac{-2Q}{m\omega^3}.$$

$$\Rightarrow x(t > \frac{\pi}{\omega}) = \left(-a - \frac{Q\pi}{m\omega^3}\right) \cos \omega t - \frac{2Q}{m\omega^3} \sin \omega t.$$

(as expected, reduces to $x(t) = -a \cos \omega t$
if $Q \rightarrow 0$, i.e. if force ~~was~~
during $0 < t < \frac{\pi}{\omega}$ vanished.)