

Lecture 17.

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Outline:

- * Energy: what is the "missing" coordinate in the Lagrangian that corresponds to energy conservation?
- * Forces of constraints: (for example how to incorporate tension needed to keep the length of the pendulum constant?).

Energy:

We know that if a coordinate is absent from the Lagrangian, then there is a conserved quantity.

If the missing coordinate is linear (e.g. x), the corresponding conserved quantity is momentum.

If the corresponding missing coordinate is an angle \Rightarrow corresponding conserved quantity is angular momentum.

Question: we know that conservative systems conserve energy. What is the "missing" coordinate in the Lagrangian, that corresponds to energy conservation?

Answer: If time, t , does not appear explicitly in the Lagrangian,

then energy is conserved.

Proof: In general, the Lagrangian depends on

$q_i(t)$, $\dot{q}_i(t)$ (q_i is a generalized coordinate) and on t :

$$\mathcal{L} = \mathcal{L}[q_i(t), \dot{q}_i(t), t].$$

therefore, the quantity \mathcal{L} will depend on t both directly and through q_i and \dot{q}_i .

$$\Rightarrow \frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i} \cdot \frac{dq_i}{dt} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial \mathcal{L}}{\partial t},$$

where $\frac{\partial \mathcal{L}}{\partial t}$ is computed at constant q_i, \dot{q}_i .

Now we make use of $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$, so that

$$\begin{aligned} \frac{d\mathcal{L}}{dt} &= \sum_i \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} + \frac{\partial \mathcal{L}}{\partial t} = \\ &= \frac{d}{dt} \left[\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right] + \frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

Here we have used Leibniz's rule $[uv]' = u'v + uv'$

We obtain thus:

$$\frac{d}{dt} \left[\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \right] = - \frac{\partial \mathcal{L}}{\partial t}.$$

\Rightarrow If \mathcal{L} does not depend explicitly on t ,

then the quantity $\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} = \text{const.}$
is conserved.

It is easy to see that

$$\sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} = E \rightarrow \text{is the energy of}$$

the system: for a particle in 3D with potential energy $V(\vec{r})$, for instance,

$$\mathcal{L} = \frac{m \vec{r}^2}{2} - V(\vec{r})$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}_i} = m \dot{r}_i \Rightarrow \sum_i \frac{\partial \mathcal{L}}{\partial \dot{r}_i} \dot{r}_i = \sum_i m \dot{r}_i^2 = m \vec{r}^2$$

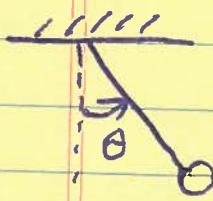
and therefore

$$\begin{aligned} \sum_i \frac{\partial \mathcal{L}}{\partial \dot{r}_i} \dot{r}_i - \mathcal{L} &= m \vec{r}^2 - \left[\frac{m \vec{r}^2}{2} - V(\vec{r}) \right] = \\ &= \frac{m \vec{r}^2}{2} + V(\vec{r}) = E_{\text{total}}. \end{aligned}$$

Forces of constraints

In the study of Lagrangian systems, we have often used knowledge of physical constraints on motion (e.g. the length of the pendulum is constant) to remove some coordinates from the problem.

For instance, the pendulum of length ℓ can be described by an "angle" coordinate θ ,



or by two coordinates

(r, θ) with the constraint $r = \ell$.

* What if we want to find the force associated with the constraint?

For example, what is the tension needed to keep the length of the pendulum constant?

In order to impose constraints at the level of Lagrangian, we use the trick of the Lagrange multipliers.

The idea is the following: let the constraint be of the form

$$f(q_i, t) = 0 \quad [\text{in the case of the pendulum} \\ f(q_i, t) = r - l].$$

Then we can impose this at the level of Lagrangian by introducing the Lagrange multiplier, λ , that is a function of time, in the Lagrangian as follows

$$\mathcal{L}^{\text{constraint}}(q_i, \dot{q}_i, t) = \mathcal{L}^{\text{Free}}(q_i, \dot{q}_i, t) + \\ + \lambda(t) \cdot f(q_i, t)$$

where $\mathcal{L}^{\text{Free}}$ is the Lagrangian without constraint.

Then we treat λ as a new degree of freedom.

Since λ does not appear in the new Lagrangian [λ constraint], the Euler-Lagrange equation associated with λ reduces

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^{\text{con}}}{\partial \dot{\lambda}} \right) = \frac{\partial \mathcal{L}^{\text{cons}}}{\partial \lambda}, \quad \frac{\partial \mathcal{L}^{\text{cons}}}{\partial \dot{\lambda}} = 0 \Rightarrow$$

$$\boxed{\frac{\partial \mathcal{L}^{\text{const}}}{\partial \lambda} = 0} \Rightarrow \text{giving precisely the constraint } f(q_i, t) = 0.$$

However, λ will now appear in the other Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}^{\text{cons}}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}^{\text{const}}}{\partial q_i}, \quad \text{i.e.} \Rightarrow$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial \mathcal{L}^{\text{Free}}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}^{\text{Free}}}{\partial q_i} + \lambda \frac{\partial f}{\partial q_i}$$

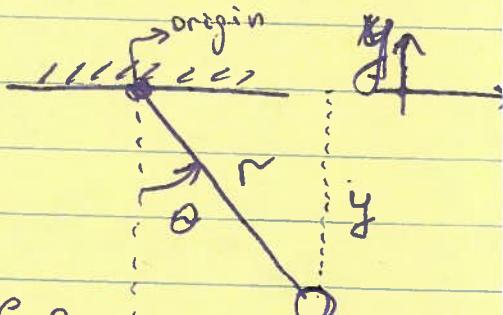
defines force
of constraint.

Let us illustrate this with a simple example.

The pendulum If we allow the length of the pendulum to vary, then its Lagrangian has the form:

$$\mathcal{L}^{\text{Free}} = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) - mg y$$

where $\begin{cases} x = r \sin \theta \\ y = -r \cos \theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta \\ \dot{y} = -\dot{r} \cos \theta + r \dot{\theta} \sin \theta \end{cases}$



Substituting these expressions into $\mathcal{L}^{\text{Free}}$ we get

$$\begin{aligned} \mathcal{L}^{\text{Free}} &= \frac{m}{2} \left[(\dot{r} \sin \theta + r \dot{\theta} \cos \theta)^2 + (-\dot{r} \cos \theta + r \dot{\theta} \sin \theta)^2 \right] \\ &\quad + m g r \cos \theta = \\ &= \frac{m}{2} [\dot{r}^2 + r^2 \dot{\theta}^2] + m g r \cos \theta \end{aligned}$$

Let us now impose the constraint $r=l$.

The constrained Lagrangian $\mathcal{L}^{\text{constraint}}$ reads

$$\mathcal{L}^{\text{const}} = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) + m g r \cos \theta + \lambda (r - l).$$

The equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}^{\text{const}}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}^{\text{const}}}{\partial q_i}$ acquire the form:

* $\dot{q}_1 = \lambda \Rightarrow r = l$

* $\dot{q}_2 = \dot{r} \Rightarrow \frac{d}{dt} [mr\dot{\theta}] = mr\dot{\theta}^2 + mg \cos \theta + \lambda$

* $\dot{q}_3 = \dot{\theta} \Rightarrow \frac{d}{dt} [mr^2\dot{\theta}] = -mg r \sin \theta$

Plugging $r = l$ in the 2nd and 3rd equations,
we obtain

* $r \Rightarrow \ddot{\theta} = ml\dot{\theta}^2 + mg \cos \theta + \lambda$

* $\theta \Rightarrow ml^2\ddot{\theta} = -mgl \sin \theta \Rightarrow \boxed{\ddot{\theta} = -\frac{g}{l} \sin \theta}$

[note: for $\theta \ll \bar{\theta} \Rightarrow \sin \theta \approx \theta \Rightarrow \ddot{\theta} = -\frac{g}{l} \theta$ - simple
harmonic oscillator]

So the θ -equation gives the same equation

we got in the standard case. {

The r -equation is now an equation for λ !

$$\lambda = -ml\dot{\theta}^2 - mg \cos \theta$$

Since λ is the force of constraint in the r -direction (it is derived from the Euler-Lagrange equations along the coordinate r), it means that there will be a tension:

Tension $= ml\dot{\theta}^2 + mg \cos\theta$ - acting on the pendulum. The first term $ml\dot{\theta}^2$ is associated to the "centrifugal" pull of the pendulum, while $mg \cos\theta$ is associated to the radial component of the gravitational force/pull.