

Outline:

- * Energy conservation - Hamiltonian formulation.
- * The phase space: how to obtain information about the evolution of our system?
- * Poisson brackets: how to find constants of motion and what is their relation to \mathcal{H} ?

Energy conservation in Hamiltonian formulation

Suppose we have a general Hamiltonian $H(p_i, q_i; t)$. H depends on time both directly and through $p_i(t)$ and $q_i(t)$. Then

$$\begin{aligned}\frac{dH}{dt} &= \sum_{i=1}^n \frac{\partial H}{\partial p_i} \cdot \dot{p}_i + \sum_{i=1}^n \frac{\partial H}{\partial q_i} \cdot \dot{q}_i + \frac{\partial H}{\partial t} \\ &= \cancel{\sum_{i=1}^n \dot{q}_i \dot{p}_i} + \cancel{\sum_{i=1}^n (-\dot{p}_i \dot{q}_i)} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t},\end{aligned}$$

where we have used $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, $\dot{q}_i = \frac{\partial H}{\partial p_i}$.

- canonical (Hamilton) equations of motion.

n - is # of degrees of freedom.

Therefore, if H does not depend on explicitly on time (i.e. $\frac{\partial H}{\partial t} = 0$), then $H = E$ is a constant of motion (conserved); so we have rediscovered energy conservation.

The phase space

For a system with n degrees of freedom, it is useful to define the phase space as $2n$ dimensional space whose coordinates are $(p_1, \dots, p_n; q_1, \dots, q_n)$. Then

all the information about the evolution of our system is given by a curve $(p_1(t), \dots, p_n(t); q_1(t), \dots, q_n(t))$ in phase space.

As an example let us consider the one-dimensional harmonic oscillator of mass m and spring constant $k = m\omega^2$. The Lagrangian of the system is:

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - V(x) = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}.$$

The momentum thus is $p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$, so that $\dot{x} = \frac{p}{m}$.

Then $H = p\dot{x} - \mathcal{L} = p\dot{x} - \left(\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2x^2 \right) =$

$$= p \cdot \left(\frac{p}{m} \right) - \frac{m}{2} \left(\left(\frac{p}{m} \right)^2 - \omega^2 x^2 \right) =$$

$$= \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \quad - \text{total energy.}$$

The equations of motion are

$$\left. \begin{array}{l} \dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \\ \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x \end{array} \right]$$

which we write as: $m\ddot{x} = -m\omega^2 x$
 $\Rightarrow \ddot{x} = -\omega^2 x.$

By choosing appropriately the origin of time
we solve it as $x(t) = A \cos \omega t,$

where A is an integration constant.

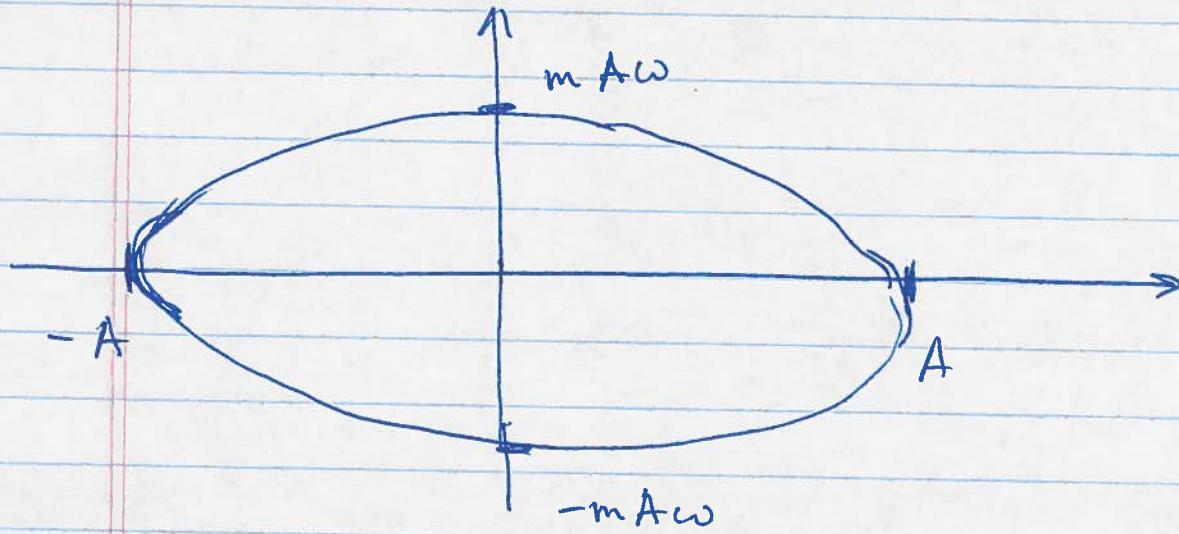
Then $p(t) = m\dot{x}(t) = -mA\omega \sin(\omega t).$

The trajectory in the phase space (P, x)

is :

$$\left\{ \begin{array}{l} x(t) = A \cos \omega t \\ P(t) = -m\omega A \sin \omega t \end{array} \right. \Rightarrow \left(\frac{x}{A} \right)^2 + \left(\frac{P}{m\omega A} \right)^2 = 1.$$

That gives an ellipse:

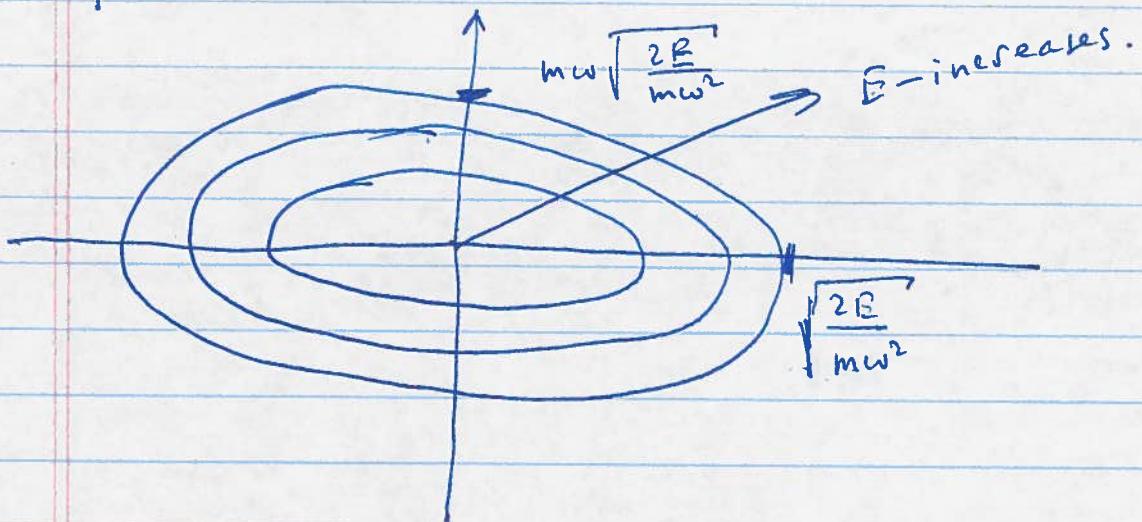


The elliptical shape could also have been obtained just by looking at energy conservation.

Indeed, for an orbit with energy E

we must have: $\frac{P^2}{2m} + \frac{m\omega^2 x^2}{2} = E$.

That describes an ellipse in the (x, p) plane, with semi-axis length $\sqrt{\frac{2E}{mw^2}}$ along the x -direction and $\sqrt{2mE} = mw\sqrt{\frac{2E}{mw^2}}$ along the p -direction



Poisson Brackets

Given two functions $f(p_i, q_i; t)$ and $g(p_i, q_i; t)$, we define the Poisson brackets of f and g as a new function of $(p_i, q_i; t)$ defined as follows:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

The Poisson brackets satisfy the following relations:

- * $\{f, g\} = -\{g, f\}$
- * $\{f, f\} = 0$
- * $\{f, c\} = 0$ if c is a constant
- * $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$
- * $\{f_1 \cdot f_2, g\} = f_1 \cdot \{f_2, g\} + \{f_1, g\} \cdot f_2$ (check this).
- * $\frac{\partial}{\partial t} \{f, g\} = \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\}$.

and the important Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

(again, this can be checked by directly by a brute force calculation).

Why are Poisson brackets important?

Suppose I would like to compute the total time derivative of $f(p_i(t), p_i'(t), \dots)$.

It reads:

$$\frac{df}{dt} = \sum_{i=1}^n \frac{\partial f}{\partial p_i} \cdot \dot{p}_i + \sum_{i=1}^n \frac{\partial f}{\partial q_i} \cdot \dot{q}_i + \frac{\partial f}{\partial t} = \text{(using Hamilton equations)}$$

$$= \sum_i \left(- \frac{\partial f}{\partial p_i} \cdot \frac{\partial H}{\partial q_i} + \frac{\partial f}{\partial q_i} \cdot \frac{\partial H}{\partial p_i} \right) + \frac{\partial f}{\partial t}$$

$$= \{f, H\} + \frac{\partial f}{\partial t} !$$

$$\text{So } \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

Therefore we see that, if f does not depend explicitly on time ($\frac{\partial f}{\partial t} = 0$), then:

$$\frac{df}{dt} = \{f, H\},$$

so that f is a constant of motion if its Poisson brackets with the Hamiltonian vanish.

No that by choosing $f=H$ we obtain

$$\frac{dH}{dt} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad (\text{since } \{H, H\} = 0),$$

i.e. that H is a constant of motion if

$$\frac{\partial H}{\partial t} = 0.$$

Another important property of Poisson

brackets is the following:

If f and g are constants of motion $\Rightarrow \{f, g\}$ is also a constant of motion.

$$\text{The proof: } \frac{d}{dt} \{f, g\} = \frac{\partial}{\partial t} \{f, g\} + \{\{f, g\}, H\} =$$

(using Jacobi identity)

$$= \left\{ \frac{\partial f}{\partial t}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} \right\} + \left\{ f, \{g, H\} \right\} + \left\{ g, \{H, f\} \right\} =$$

$$= \left\{ \frac{\partial f}{\partial t} + \{f, H\}, g \right\} + \left\{ f, \frac{\partial g}{\partial t} + \{g, H\} \right\} =$$

$$= \left\{ \frac{df}{dt}, g \right\} + \left\{ f, \frac{dg}{dt} \right\} = 0.$$

" 0

As a consequence, if we have found two constants of motion, then their Poisson brackets is also a constant of motion!