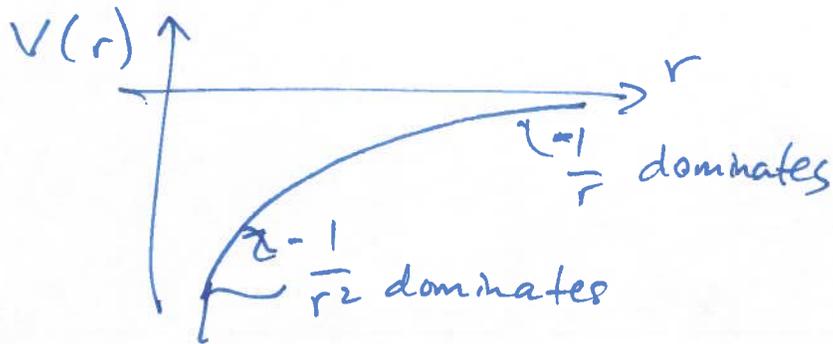


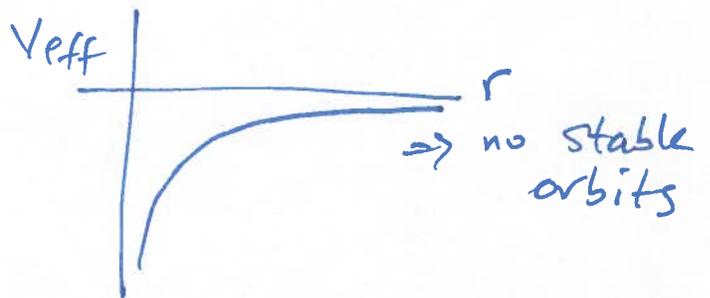
$$(1. (a) V(r) = -\int F(r) dr = \int \left(\frac{\alpha}{r^2} + \frac{\beta}{r^3} \right) dr = -\frac{\alpha}{r} + \frac{-\beta}{2r^2} + \text{const.}$$

Set const. = 0 st. $V=0$ @ infinity.

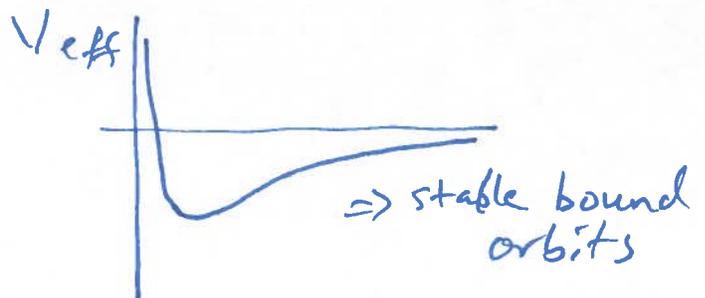


$$V_{\text{eff}} = -\frac{\alpha}{r} - \frac{\beta}{2r^2} + \frac{l^2}{2mr^2} = -\frac{\alpha}{r} + \left(\frac{l^2}{m} - \beta \right) \cdot \frac{1}{2r^2}$$

(b) For $\frac{l^2}{m} - \beta \leq 0$



For $\frac{l^2}{m} - \beta > 0$



So need $l > \sqrt{\beta m}$

For these values of l :

r_c , radius of ~~orbit~~ circular orbit must satisfy

$$V'_{\text{eff}}(r_c) = 0.$$

$$\text{i.e. } \frac{\alpha}{r_c^2} - \left(\frac{l^2}{m} - \beta\right) \frac{1}{r_c^3} = 0 \Rightarrow r_c = \frac{\left(\frac{l^2}{m} - \beta\right)}{\alpha}$$

The corresponding energy is

$$E_c = -\frac{\alpha}{r_c} + \left(\frac{l^2}{m} - \beta\right) \frac{1}{2r_c^2}$$

$$= \frac{-\alpha^2}{\left(\frac{l^2}{m} - \beta\right)} + \left(\frac{l^2}{m} - \beta\right) \cdot \frac{1}{2} \frac{\alpha^2}{\left(\frac{l^2}{m} - \beta\right)^2}$$

$$= -\frac{1}{2} \frac{\alpha^2}{\left(\frac{l^2}{m} - \beta\right)} < 0$$

(c) Bound orbits for $-\frac{\alpha^2}{2\left(\frac{l^2}{m} - \beta\right)} < E < 0$.

r_{\min} and r_{\max} are found by solving

$$V_{\text{eff}}(r_{\min}, r_{\max}) = E, \quad \text{i.e. } -\frac{\alpha}{r} + \left(\frac{l^2}{m} - \beta\right) \frac{1}{2r^2} = E$$

Giving, $E r^2 + \alpha r - \frac{1}{2} \left(\frac{l^2}{m} - \beta\right) = 0$

$$r = \frac{1}{2E} \left[-\alpha \pm \sqrt{\alpha^2 + 2E \left(\frac{l^2}{m} - \beta\right)} \right]$$

or, remembering that $E < 0$,

$$r = \frac{1}{2|E|} \left[\alpha \pm \sqrt{\alpha^2 - 2|E| \left(\frac{l^2}{m} - \beta\right)} \right].$$

(d) Use the equation

$$\begin{aligned}\frac{d^2 u}{d\theta^2} + u &= -\frac{m}{l^2 u^2} F\left(\frac{1}{u}\right) = -\frac{m}{l^2 u^2} [-\alpha u^2 - \beta u^3] \\ &= \frac{m\alpha}{l^2} + \frac{m\beta}{l^2} u\end{aligned}$$

i.e. $\frac{d^2 u}{d\theta^2} + \left(1 - \frac{m\beta}{l^2}\right) u = \frac{m\alpha}{l^2}$

Denote by $\gamma^2 > 0$ the quantity $\gamma^2 = 1 - \frac{m\beta}{l^2}$
(see result for bound orbits in (a))

Then the equation is that of a shifted harmonic oscillator:

$$\frac{d^2 u}{d\theta^2} + \gamma^2 u = \frac{\alpha m}{l^2}$$

Redefine $u = \bar{u} + u_0$, with u_0 constant chosen to cancel the term $\alpha m / l^2$:

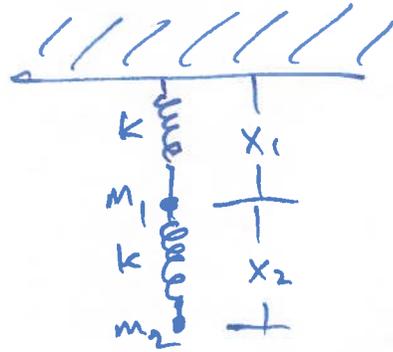
$$\frac{d^2 \bar{u}}{d\theta^2} + \gamma^2 \bar{u} + \gamma^2 u_0 = \frac{\alpha m}{l^2} \Rightarrow \text{choose } u_0 = \frac{\alpha m}{l^2 \gamma^2}$$

Then $\frac{d^2 \bar{u}}{d\theta^2} + \gamma^2 \bar{u} = 0 \Rightarrow \bar{u} = A \cos(\gamma(\theta + \theta_0))$

implying $u = A \cos[\gamma(\theta + \theta_0)] + \frac{\alpha m}{l^2 \gamma^2}$

or $r(\theta) = \frac{1}{A \cos\left[\sqrt{1 - \frac{m\beta}{l^2}}(\theta + \theta_0)\right] + \frac{\alpha m}{l^2} \cdot \frac{1}{1 - \frac{m\beta}{l^2}}}$

2. (a)



2 degrees
of freedom

$$T = \frac{m}{2} \dot{x}_1^2 + \frac{m}{2} (\dot{x}_1 + \dot{x}_2)^2$$

$$V_{\text{springs}} = \frac{k}{2} x_1^2 + \frac{k}{2} x_2^2$$

$$V_{\text{gravity}} = -mgx_1 - mg(x_1 + x_2)$$

$$L = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_1^2 + 2\dot{x}_1\dot{x}_2 + \dot{x}_2^2) - \frac{k}{2} (x_1^2 + x_2^2) + mg(2x_1 + x_2)$$

Equations of motion

$$\begin{cases} 2m\ddot{x}_1 + m\ddot{x}_2 = -kx_1 + 2mg \\ m\ddot{x}_1 + m\ddot{x}_2 = -kx_2 + mg \end{cases}$$

Defining $\omega^2 = k/m$, these equations are equivalent to

$$\begin{cases} 2\ddot{x}_1 + \ddot{x}_2 = -\omega^2 x_1 + 2g \\ \ddot{x}_1 + \ddot{x}_2 = -\omega^2 x_2 + g \end{cases}$$

(b) Equilibrium positions

$$-w^2 x_1 + 2g = 0 \Rightarrow x_1 = \frac{2g}{w^2} = \frac{2mg}{k}$$

$$-w^2 x_2 + g = 0 \Rightarrow x_2 = \frac{g}{w^2} = \frac{mg}{k}$$

Stable? Yes, since $V = \frac{k}{2}(x_1^2 + x_2^2) - mg(2x_1 + x_2)$

$$\text{and } \frac{\partial^2 V}{\partial x_1 \partial x_2} \geq 0.$$

3. (a) Equation of motion

$$F = ma$$

$$-b\sqrt{v} = m \frac{dv}{dt} \quad (v \text{ positive})$$

$$\frac{dv}{\sqrt{v}} = -\frac{b}{m} dt$$

Integrate:

$$2(\sqrt{v(t)} - \sqrt{v(0)}) = -\frac{b}{m} t$$

$$\sqrt{v(t)} = \sqrt{v(0)} - \frac{bt}{2m}$$

$$v(t) = \left(\sqrt{v_0} - \frac{bt}{2m} \right)^2.$$

(b) So the particle stops @ $v(t_{\text{stop}}) = 0$, i.e.

$$\sqrt{v_0} = \frac{bt_{\text{stop}}}{2m} \Rightarrow t_{\text{stop}} = \frac{2m\sqrt{v_0}}{b}.$$

$$(c) \frac{dx}{dt} = v(t) = \left(\sqrt{v_0} - \frac{bt}{2m} \right)^2$$

$$x(t) = x(0) = \int_0^t \left(\sqrt{v_0} - \frac{bt'}{2m} \right)^2 dt'$$

To compute the integral, substitute

$$y = \sqrt{v_0} - \frac{bt'}{2m} \rightarrow t' = \frac{2m}{b}(\sqrt{v_0} - y) \rightarrow dt' = -\frac{2m}{b} dy$$

Also, $y = \sqrt{v_0}$ for $t' = 0$

$$y = \sqrt{v_0} - \frac{bt}{2m} \text{ for } t' = t$$

$$\text{Therefore } x(t) = \int_{\sqrt{v_0}}^{\sqrt{v_0} - \frac{bt}{2m}} \left(-\frac{2m}{b} \right) y^2 dy = \frac{2m}{b} \frac{y^3}{3} \Bigg|_{\sqrt{v_0} - \frac{bt}{2m}}^{\sqrt{v_0}}$$

$$= \frac{2m}{3b} \left(\sqrt{v_0}^3 - \left[\sqrt{v_0} - \frac{bt}{2m} \right]^3 \right)$$

When the particle stops,

$$x(t_{\text{stop}}) = \frac{2m}{3b} v_0^{3/2}.$$

(d) Dimensions: $[m] = [M]$

$$[v_0] = [LT^{-1}]$$

$$[b] = ? \quad b\sqrt{v} \text{ is a force} = [MLT^{-2}]$$

$$[b][L^{1/2}T^{-1/2}] = [MLT^{-2}]$$

$$[b] = [ML^{1/2}T^{-3/2}]$$

How do I construct a length out of m , v_0 , and b ?

$$\begin{aligned} [L] &= m^\alpha v_0^\beta b^\gamma = [M^\alpha][L^\beta T^{-\beta}][M^\gamma L^{\gamma/2} T^{-3/2\gamma}] \\ &= [M^{\alpha+\gamma} L^{\beta+\frac{\gamma}{2}} T^{-\beta-\frac{3}{2}\gamma}] \end{aligned}$$

So need

$$\alpha + \gamma = 0 \quad \rightarrow \alpha = -\gamma$$

$$\beta + \frac{\gamma}{2} = 1 \quad \rightarrow \beta = 1 - \frac{\gamma}{2}$$

$$-\beta - \frac{3}{2}\gamma = 0 \quad \rightarrow -1 + \frac{\gamma}{2} - \frac{3\gamma}{2} = 0$$

$$= -1 - \gamma = 0 \Rightarrow \gamma = -1$$

$$\therefore \beta = \frac{3}{2} \text{ and } \alpha = 1$$

$$\text{So } x(t_{\text{stop}}) \approx m v_0^{3/2} b^{-1} \approx \frac{m}{b} v_0^{3/2}$$