PHY 421 FALL 2015 - MIDTERM EXAM 2

November 23rd, 2015

Solve the following three problems. All the problems carry equal credit (but the questions inside each problem can have different weight). Books and notes are allowed.

1. A particle of mass m is subject to a central force $\vec{F}(\vec{r}) = -k\hat{r}$, with k a *positive* constant with appropriate dimensions and $\hat{r}$ denoting the radially directed outgoing unit vector.

(a) Express the force $\vec{F}(\vec{r})$ in Cartesian coordinates, i.e., decomposing it as $\vec{F} = F_x(x, y, x)\hat{\imath} + F_y(x, y, x)\hat{\jmath} + F_z(x, y, x)\hat{k}$.

(b) The particle has angular momentum $\vec{\ell}$. Compute the potential $V(r)$ (choosing the integration constant such that $V(r = 0) = 0$) and the effective potential $V_{\text{eff}}(r)$ associated with $\vec{F}(\vec{r})$.

(c) Find, as a function of k, $\ell \equiv |\vec{\ell}|$, and m, the radius of the circular orbit for the particle. Call this r_c. What is, as a function of k, ℓ, and m, the velocity of the particle in such an orbit? (recall: $\dot{\theta} = \ell/(mr^2)$). What is the total energy $E_c = V_{\text{eff}}(r_c)$ of the particle on this orbit?

(d) For energies much larger than E_c but with the particle still in an orbit, find, as a function of E, the approximate value of the velocity $|\vec{v}|$ of the particle at the aphelion (i.e., at its *maximal* distance from the center) and at the perihelion (i.e., at its *minimal* distance from the center). [Hint to proceed: At small r, one term in the effective potential dominates, and the energy E can be approximated as consisting only of this term. For large r, again one term in $V_{\text{eff}}(r)$ dominates; etc.].

2. Consider a *one-dimensional* system of two particles with masses m_1 and m_2 moving towards each other with velocities v_1 and v_2, with $v_1 > 0$ and $v_2 < 0$. At some point the two particles scatter and then leave with velocities v_1' and v_2'. During the scattering, however, *the two particles exchange part of their mass*, so that the outgoing particles have masses m_1' and m_2', while the total kinetic energy is conserved. Assume that the total mass is conserved ($m_1' + m_2' = m_1 + m_2$).

(a) Write the equations of conservation of energy and momentum for this system, and then write them in terms of the velocity of the center of mass of the system (V before the scattering and V' after the scattering) and of the relative velocity of the two particles (v before the scattering and v' after the scattering).

(b) Solve the equations of conservation of momentum and energy found above, writing the final velocities V' and v' as a function of V and v as well as of m_1, m_2, m_1', and m_2'.

(c) How does your result change if the total mass is not conserved, i.e., if $m_1' + m_2' \neq m_1 + m_2$?

3. A particle of mass m is subject to a central force field with potential $V(r) = -\alpha/r - \alpha r_0^2/r^3$, where α and r_0 are both *positive* constants.

(a) Sketch a plot of $V(r)$.

(b) For which values of the angular momentum ℓ does this system admit circular orbits? Compute the radius of the possible circular orbits and determine whether such orbits are stable or unstable.

(c) Give an approximate expression for the radius of the circular orbits for small values of r_0. Determine also with respect to which quantity having dimensions of a length should r_0 be small.

(d) **[Bonus: 3 marks]** In the regime of validity of the approximation of (c), determine approximately the expression of $u(\theta)$ (remember that $u(\theta) = 1/r(\theta)$) for an orbit near the *unstable* circular orbit.