

The fundamental Poisson brackets are ~1- especially important:

$$\{q_a, q_b\} = \sum_i \frac{\partial q_a}{\partial q_i} \overset{q^0}{\cancel{\frac{\partial q_b}{\partial p_i}}} - \overset{p^0}{\cancel{\frac{\partial q_a}{\partial p_i}}} \frac{\partial q_b}{\partial q_i} = 0$$

$$\{p_a, p_b\} = \sum_i \frac{\partial p_a}{\partial q_i} \overset{p^0}{\cancel{\frac{\partial p_b}{\partial p_i}}} - \overset{q^0}{\cancel{\frac{\partial p_a}{\partial p_i}}} \frac{\partial p_b}{\partial q_i} = 0$$

$$\{q^0, p_b\} = \sum_i \frac{\partial q^0}{\partial q_i} \frac{\partial p_b}{\partial p_i} - \overset{p^0}{\cancel{\frac{\partial q^0}{\partial p_i}}} \frac{\partial p_b}{\partial p_i} = \\ = \sum_i \delta_{ai} \delta_{bi} = \delta_{ab}$$

Using these properties, we can find for example that if two components of the angular momentum, say  $l_x$  and  $l_y$ , are conserved  $\Rightarrow$  then also  $l_z$  is conserved.

Proof:

$$\begin{aligned} \{l_x, l_y\} &= \{y p_z - z p_y, z p_x - x p_z\} = \\ &= \{y p_z, z p_x\} - \{y p_z, x p_z\} - \{z p_y, z p_x\} + \{z p_y, x p_z\} \\ &= y \{p_z, z p_x\} + \{y, z p_x\} p_z - y \{p_z, x p_z\} - \{y, x p_z\} p_z \\ &\quad - z \{p_y, z p_x\} - \{z, z p_x\} p_y + z \{p_y, x p_z\} + \{z, x p_z\} p_y \\ &= (\text{keeping only terms of the form } \{x_i, p_j\}) \end{aligned}$$

$$= y \{ p_z, z \} p_x + x \{ z, p_z \} p_y = -y p_x + x p_y = \underline{\underline{l}_z}.$$

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An important role of Poisson's brackets is that they are useful in the transition from classical mechanics to quantum mechanics.

In fact, canonical quantization of a system is ~~realized by~~ realized by promoting coordinates, momenta, and all other quantities to operators. These operators in general do not commute, and their commutation relations are obtained by promoting Poisson brackets to commutators, and multiplying them by  $i\hbar$ . So for example

from  $\{ q_a, p_b \} = \delta_{ab}$  in Classical Mechanics

we obtain

$[\hat{q}_a, \hat{p}_b] = i\hbar \delta_{ab}$  in Quantum Mechanics.

In particular

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from  $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$

we derive the quantum mechanical law  
of evolution of operators in Heisenberg  
picture:

$$\frac{df}{dt} = \frac{[f, H]}{i\hbar} + \frac{\partial f}{\partial t} = \frac{i}{\hbar} [H, f] + \frac{\partial f}{\partial t}.$$