

## Mid-term 1

### Solutions

Problem 1:

$$f = f(r), \text{ where } r = \sqrt{x^2 + y^2 + z^2}, \quad \vec{r} = \vec{i}x + \vec{j}y + \vec{k}z; \quad r = |\vec{r}|$$

$$\begin{aligned} \Rightarrow \underbrace{\vec{\text{grad}} f(r)}_{\vec{r}} &= \vec{i} \frac{\partial f(r)}{\partial x} + \vec{j} \frac{\partial f(r)}{\partial y} + \vec{k} \frac{\partial f(r)}{\partial z} = \\ &= \vec{i} \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial x} + \vec{j} \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial y} + \vec{k} \frac{\partial f(r)}{\partial r} \cdot \frac{\partial r}{\partial z} = \\ &= \left( \frac{\vec{i}x}{\sqrt{x^2 + y^2 + z^2}} + \frac{\vec{j}y}{\sqrt{x^2 + y^2 + z^2}} + \frac{\vec{k}z}{\sqrt{x^2 + y^2 + z^2}} \right) \frac{\partial f(r)}{\partial r} = \\ &= \frac{\vec{i}x + \vec{j}y + \vec{k}z}{r} \cdot \frac{\partial f(r)}{\partial r} = \underbrace{\frac{\vec{r}}{r} \cdot \frac{\partial f(r)}{\partial r}}_v \end{aligned}$$

$$\text{Here we used } \frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

and analogous expressions for  $\frac{\partial r}{\partial y}$  and  $\frac{\partial r}{\partial z}$ .

Problem 2: We have  $U(x) = U_0 \left( \frac{a}{x} + \frac{x}{a} \right)$

for  $x > 0$ . Equilibrium points are defined by

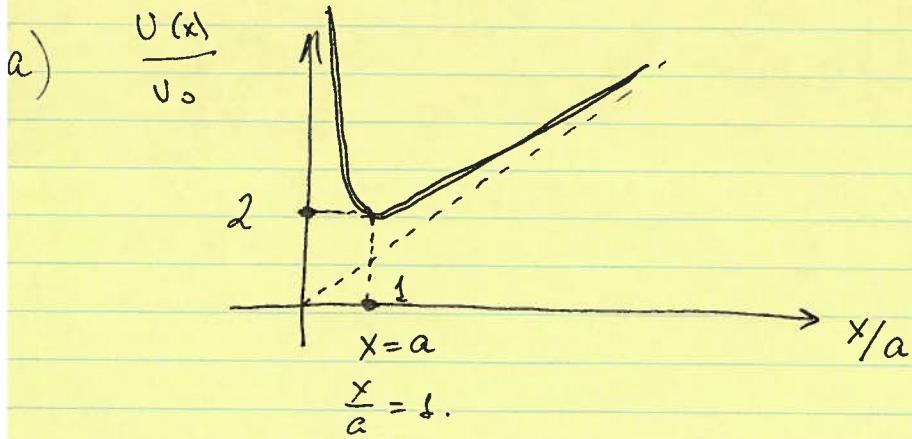
$\frac{dU}{dx} = 0$ . The stability of equilibrium is deter-

mined by  $\frac{d^2U}{dx^2}$ , at those points.

$\Rightarrow \frac{dU}{dx} = U_0 \left( -\frac{a}{x^2} + \frac{1}{a} \right)$  which vanishes at  $x=a$ .

Now evaluate  $\frac{d^2U}{dx^2} \Big|_{x=a} = \frac{2U_0 a}{x^3} \Big|_{x=a} = \frac{2U_0}{a^2} > 0$ !

This indicates that the equilibrium point is stable. It corresponds to the minimum of  $U(x)$ .



b)  $U(x=a) = 2U_0 = 2J$

c)  $F = -\frac{\partial U}{\partial x} = -U_0 \left( -\frac{a}{x^2} + \frac{1}{a} \right) = \frac{U_0 a}{x^2} - \frac{U_0}{a}$ .

Problem 3: The oscillation is underdamped  $\Rightarrow$

$$\Rightarrow x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta),$$

where  $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$  is the angular frequency of the damped oscillator, and  $\omega_0$  is the angular frequency in the absence of damping.

$x(t=0) = A$

at  $t = 4T = \frac{8\pi}{\omega_1}$  is the time corresponding to 4 cycles.

$$x(4T) = A e^{-\beta \frac{8\pi}{\omega_1}} \cos(8\pi - \delta)$$

The amplitude

So we have  $\frac{A e^{-\frac{8\pi\beta}{\omega_1}}}{A} = \frac{1}{e} \Rightarrow \frac{8\pi\beta}{\omega_1} = 1.$

and since  $\beta = \sqrt{\omega_0^2 - \omega_1^2}$ , we have

$$1 = \frac{(8\pi)^2 \beta^2}{\omega_1^2} = \frac{(8\pi)^2 (\omega_0^2 - \omega_1^2)}{\omega_1^2}$$

$$\omega_1^2 + (8\pi)^2 \omega_1^2 = + \omega_0^2 \cdot (8\pi)^2$$

$$\frac{\omega_1^2}{\omega_0^2} = \frac{(8\pi)^2}{64\pi^2 + 1} \Rightarrow \boxed{\frac{\omega_1}{\omega_0} = \frac{8\pi}{\sqrt{64\pi^2 + 1}}}.$$