Week 1: Lectures 1 & 2.

Outline:

* Second quantization
* Canonical transformation
* Quasiparticles
* Examples: fermionic chain
* Jordan-Wigner transformation
* 1D Heisenberg spin-½ chain
* XXZ chain.

* Quasiparticles: 1. second quantization; Canonical transformat

The systems with large number of identical particles are convenient to study using the second quantization method.

a) Consider a system of bose-particles, each of which can find itself in one of the following states:

$$\psi_1(x),\ \psi_2(x),\ \psi_3(x),\ \dots \equiv \psi_i(x),\quad i=1,2,\dots$$

Many-body wavefunction is thus given by occupation numbers, which show how many particles occupy the given state $\psi_i(x)$. In Dirac's notations such states can be written as

$$|\dots, N_{i-1}, N_i, N_{i+1}, \dots\rangle, \quad \text{where}$$

occupation numbers N_i acquire arbitrary positive integer numbers $N_i = 0, 1, 2, \dots$

Canonical creation and annihilation operators

a_i^+, a_i are introduced as follows:

$$a_i |\dots N_i, N_{i+1}, \dots\rangle = \sqrt{N_i}\, |\dots, N_i - 1, N_{i+1} \dots\rangle$$

$$a_i^+ |\dots N_i, N_{i+1}, \dots\rangle = \sqrt{N_i + 1}\, |\dots, N_i + 1, N_{i+1}, \dots\rangle$$

From this definition, one can easily compute commutation relations of creation and annihilation operators:

$$* \quad [a_i, a_j^+] = a_i a_j^+ - a_j^+ a_i = \delta_{ij}$$

<u>proof:</u> Let $i \neq j \Rightarrow a_i a_j^+ | \dots N_i \dots N_j \dots \rangle =$

$$= \sqrt{N_i} \sqrt{N_j + 1} \, | \dots N_i - 1 \dots N_j + 1 \dots \rangle$$

and $\quad a_j^+ a_i | \dots N_i \dots N_j \dots \rangle = \sqrt{N_j + 1} \sqrt{N_i} \times$

$$\times | \dots N_i - 1 \dots N_j + 1 \dots \rangle$$

$$\Rightarrow (a_i a_j^+ - a_j^+ a_i) | \dots N_i \dots N_j \dots \rangle = 0 \Rightarrow [a_i a_j^+] = 0$$

if $i \neq j$.

If $i = j$

$$(a_i a_i^+ - a_i^+ a_i) | \dots N_i \dots \rangle =$$

$$= \sqrt{N_i + 1} \cdot \sqrt{N_i + 1} \, | \dots N_i \dots \rangle -$$

$$- \sqrt{N_i} \cdot \sqrt{N_i} \, | \dots N_i \dots \rangle = [N_i + 1 - N_i] \, | \dots N_i \dots \rangle$$

$$= | \dots N_i \dots \rangle$$

$$\Rightarrow (a_i a_i^+ - a_i^+ a_i) = \mathbb{1} \rightarrow \text{identity operator.}$$

* Similarly, one can show that

$$[a_i, a_j] = [a_i^+, a_j^+] = 0.$$

As the next step, one defines a

$\hat{\psi}$-operator : creation/annihilation operator of boson at point $\underline{x}$:

$$\hat{\Psi}(x) = \sum_i \hat{a}_i \, \psi_i(x) \quad ; \quad \hat{\psi}^+(x) = \sum_i \hat{a}_i^+ \psi_i^*(x).$$

Functions $\psi_i(x)$, $i = 1, 2, \ldots$ are chosen to compose a full <u>orthonormal</u> set, i.e.,

$$\int \psi_i(x)\, \psi_j(x)\, dx = 0 \;, \quad i \neq j$$

$$\| \psi_i(x) \| = \left[\int | \psi(x) |^2 dx \right]^{1/2} = 1$$

From here, one can obtain commutation relations of $\hat{\psi}$-operators as follows

$$[\hat{\psi}(x), \hat{\psi}^+(x')] = \delta(x - x')$$

$$[\hat{\psi}(x), \hat{\psi}(x')] = [\hat{\psi}^+(x), \hat{\psi}^+(x')] = 0$$

b) Consider a system of <u>Fermi-statistics</u>. Then the formal definitions of occupation numbers and creation/annihilation operators are similar. The main difference follows from Pauli's principle, which implies that $\underline{N_i = 0, 1}$ - only.

Therefore, canonical operators a_i^+, a_i act as follows:

$$a_i \,| \ldots N_i, N_{i+1}, \ldots \rangle = \begin{cases} |\ldots 0, N_{i+1} \ldots \rangle, & \text{if } N_i = 1 \\ 0, & \text{if } N_i = 0 \end{cases}$$

$$a_i^+ \,| \ldots N_i, N_{i+1}, \ldots \rangle = \begin{cases} 0, & \text{if } N_i = 1 \\ |0, \ldots 1, N_{i+1}, \ldots \rangle, & \text{if } N_i = 0 \end{cases}$$

Secondly, the asymmetry of many-particle state with respect to exchange of particles, implies anticommutativity of a_i, a_i^+:

$$\{ a_i^+, a_j \}_+ = a_i^+ a_j + a_j a_i^+ = \delta_{ij}$$

$$\{ a_i, a_j \}_+ = \{ a_i^+, a_j^+ \} = 0 .$$

Anticommutativity of $\hat{\psi}$-operators can be cast as follows:

$$\{ \hat{\psi}(x), \hat{\psi}^+(x') \} = \delta(x - x'), \quad \{ \hat{\psi}(x), \hat{\psi}(x') \} = \{ \hat{\psi}^+(x), \hat{\psi}^+(x') \} = 0.$$

Hamiltonian of a many-body system.

Second quantization establishes convenient "language" between single- and many-particle systems.

For example a system of non-interacting bosons or fermions ~~can be described~~ moving in a potential $U(r)$, can be described by the following Hamiltonian

$$\hat{H}_0 = \int \left[-\frac{\hbar^2}{2m} \hat{\psi}^{+}(r) \nabla^2 \hat{\psi}(r) + \hat{\psi}^{+}(r) \hat{\psi}(r) U(r) \right] d^3 r$$

Particles here are assumed to be non-relativistic, having a quadratic dispersion.

If these particles interact via $V(\vec{r}_1 - \vec{r}_2)$ potential then the Hamiltonian has to be supplemented by

$$\hat{H}_{int} = \frac{1}{2} \iint \hat{\psi}^{+}(\vec{r}_1) \psi^{+}(\vec{r}_2) V(\vec{r}_1 - \vec{r}_2) \hat{\psi}(\vec{r}_2) \hat{\psi}(\vec{r}_1) d^3 r_1 d^3 r_2$$

The secondary quantized density operator

$\hat{\rho}(\vec{r}) = \hat{\psi}^{+}(r) \hat{\psi}(r)$ is a many-body equivalent

of single-particle probability density $|\psi(\vec{r})|^2$.

$$\hat{N} = \int_{Vol.} \hat{\psi}^+(\vec{r})\,\hat{\psi}(\vec{r})\,d^3r \quad \text{is the particle number}$$

operator. So far, in this section, ~~[illegible]~~ (of $\hat{H}_0$, $\hat{H}_{int}$, $\hat{\rho}(r)$, $\hat{N}$) all the expressions are the same for bosons and fermions.

Canonical transformations in second quantization

Recall, that in classical mechanics canonical transformations of phase space $(p,q) \to (p',q')$ are introduced via Poisson brackets, in such a way, that they preserve Hamilton's equations of motion:

$$\dot{p} = \{p, H\}, \quad \dot{q} = \{q, H\}.$$

In quantum mechanics one promotes Poisson brackets to commutators (for example in Heisenberg's scheme equations of motion acquire the form $i\hbar\partial_t \hat{A} = [\hat{A}, \hat{H}]$).

So canonical transformations by definition preserve commutation relations of operators.

Similarly to classical mechanics, canonical transformations are important, as they preserve the form of equations of motion.

By choosing a canonical transformation, one may transition from interacting particles to non-interacting quasi-particles.

Often, one considers <u>linear</u> canonical transformations (of bosons or fermions):

$$\begin{aligned} \bar{a}_i &= \sum_j \left(U_{ij}\, a_j + V_{ij}\, a_j^+ \right) \\ \bar{a}_i^+ &= \sum_j \left(V_{ij}^*\, a_j + U_{ij}^*\, a_j^+ \right) \end{aligned} \qquad (X)$$

- called Bogolubov transformation.

One of the typical problems in many-body theory is to find the spectrum of eigenstates of the Hamiltonian, if ~~the~~ it is quadratic in particle creation annihilation operators. It is easy to check that $\forall$ such Hamiltonian can be diagonalized via Bogolubov's transformation, s.t.

$$\hat{H} = \sum_{ij} h_{ij}^{(1)} a_i^+ a_j + h_{ij}^{(2)} a_i a_j + h.c. =$$

$$= \sum_i \underset{\substack{\downarrow \\ \text{quasiparticle spectrum.}}}{\varepsilon_i}\, \bar{a}_i^+ \bar{a}_j + \underbrace{\langle 0 | \hat{H} | 0 \rangle}_{\text{zero point energy.}}$$

Transformation (X) is canonical, if it preserves commutation relations:

$$[a_i, a_j] = 0, \qquad [a_i, a_j^+] = \delta_{ij} \qquad \text{- for bosons}$$

$$\{a_i, a_j\} = 0, \qquad \{a_i, a_j^+\} = \delta_{ij} \qquad \text{- for fermions.}$$

or, we can write

$$[a_i, a_j]_\pm = 0, \qquad [a_i, a_j^+]_\pm = \delta_{ij}.$$

These conditions imply that U and V matrices should obey:

$$U_{ki} V_{kj} \pm V_{ki} U_{kj} = 0$$

$$U^*_{ki} U_{kj} \pm V^*_{ki} V_{kj} = \delta_{ij},$$

where $\underline{+}$ sign corresponds to <u>fermions</u>,

$\underline{\underline{-}}$ sign corresponds to <u>bosons</u>.

Canonical transformation (X) for <u>bose-operators</u> is seen already for a state of single boson (e.g., quantum-mechanical oscillator).

In this case, U and V -are just numbers (and not matrices). Then there are only 2 types of ~~a~~ homogeneous canonical transformations:

$$\begin{cases} \bar{a} = ch\lambda \cdot a + sh\lambda \cdot a^+ \\ \bar{a}^+ = sh\lambda \cdot a + ch\lambda \cdot a^+ \end{cases} \quad \text{or} \quad \begin{cases} \bar{a} = e^{i\varphi}\, a \\ \bar{a}^+ = e^{-i\varphi}\, a \end{cases}$$

where λ and φ are real parameters.
More general transformations are just compositions (direct products) of these transformations.

In case of <u>fermi-statistics</u>, canonical transformations are defined in a similar way. For one single fermion degree of freedom, all canonical transformations amount to ~~a~~ taking a conjugate; and multiplication by a phase

$$\tilde{a} = e^{i\varphi} a^+, \quad \tilde{a}^+ = e^{-i\varphi} a.$$

More general transformations in this case are nonlinear.

More general <u>linear</u> transformations appear in a system of 2 fermions:

$$\tilde{a} = \cos\theta\, a - \sin\theta\, b^+, \quad a^+ = \cos\theta \cdot a^+ - \sin\theta \cdot b$$

$$\tilde{b}^+ = \sin\theta \cdot a + \cos\theta \cdot b^+, \quad \tilde{b} = \sin\theta \cdot a^+ + \cos\theta \cdot b,$$

where θ is a parameter.

Interestingly, the first canonical transformation for bosons is "pseudo"-euclidean rotation = Lorenz transformation with rapidity λ. in 2D space-time.

For 2 fermions, however, one obtains rotation of euclidean space.

Canonical transformations are not only linear and homogeneous. Sometimes nonlinear inhomogeneous linear transformations are quite useful.

Fourier transformation is a type of linear canonical transformations. For ψ-operators one has

$$\hat{\psi}(\vec{r}) = \int e^{i\vec{p}\vec{r}}\, \hat{a}_{\vec{p}} \frac{d^3p}{(2\pi)^3}, \quad \hat{\psi}^+(\vec{r}) = \int e^{-i\vec{p}\vec{r}}\, \hat{a}^+_{\vec{p}} \frac{d^3p}{(2\pi)^3}.$$

One normalizes $\hat{a}_{\vec{p}}$, $\hat{a}^{+}_{\vec{p}}$ in such a way that

$$\left[\hat{a}_{\vec{p}}, \hat{a}^{+}_{\vec{p}}\right]_{\pm} = (2\pi)^3\, \delta^{(3)}(\vec{p}_1 - \vec{p}_2).$$

An important class of problems in solid state physics is given by $\hat{\Psi}$-operators defined on lattice sites. Then Fourier transform of $\hat{\Psi}$ can be defined in a way that $\vec{r}$ belongs to lattice sites and $\vec{p}$ lives in a Brillouin zone.

In 1D: $r_n = an$, and $-\pi/a < p < \pi/a$ is the Brillouin zone.

<u>Example:</u> Fermionic chain is given by

$$\hat{H} = \sum_{i=-\infty}^{\infty} \Big(J_1 c_i^+ c_{i+1} + J_1 c_{i+1}^+ c_i + J_2 c_i c_{i+1} + J_2 c_{i+1}^+ c_i^+ - 2\mu\, c_i^+ c_i \Big).$$

This Hamiltonian can be diagonalized using Bogolubov transformation.

Solution 1: Let us Fourier transform our creation-annihilation operators:

$$c_m = \int_{-\pi}^{\pi} e^{ikm} c_k \frac{dk}{2\pi}.$$ Then

$$\sum_m c_m^+ c_{m+1} = \sum_k c_k^+ c_k e^{ik}$$

$$\sum_m c_m c_{m+1} = \sum_k c_k c_{-k} e^{-ik}$$

$$\sum_m \left(c_m^+ c_{m+1} + c_{m+1}^+ c_m\right) = \sum_k 2\cos k \, c_k^+ c_k,$$ where

$$\sum_k \equiv \int_{-\pi}^{\pi} \cdots \frac{dk}{2\pi}.$$ Because of anti-commutativity of c_k and c_{-k}

$$\sum_m c_m c_{m+1} = -i \sum_k \sin k \, c_k c_{-k}.$$

After this, our Hamiltonian becomes

$$\hat{H} = \sum_k \Big[2(J_1 \cos k - J) c_k^+ c_k - iJ_2 \sin k \, c_k c_{-k} + iJ_2 \sin k \, c_{-k}^+ c_k^+$$

One can perform a rotation: $c_k = e^{i\pi/4} b_k$,

$$c_k^+ = e^{-i\pi/4} b_k^+,$$

~~for m~~ so that the Hamiltonian becomes real: $$\hat{H} = \sum_k \left[(J_1 \cos k - \mu) b_k^+ b_k + J_2 \sin k \, b_k b_{-k} + h.c. \right]$$

Let us look for fermionic Bogolubov transformation in the form

$$b_k = u_k \bar{c}_k + v_k \bar{c}_{-k}^+ , \qquad b_k^+ = u_k \bar{c}_k^+ + v_k \bar{c}_{-k}$$

$$b_{-k}^+ = -v_k \bar{c}_k + u_k \bar{c}_{-k}^+ , \qquad b_{-k} = -v_k \bar{c}_k^+ + u_k \bar{c}_{-k} ,$$

with real u_k and v_k, where $u_k^2 + v_k^2 = 1$.

This gives us

$$\hat{H} = \sum_k (J_1 \cos k - \mu) \Big(u_k^2 \bar{c}_k^+ \bar{c}_k + u_k v_k (\bar{c}_k^+ \bar{c}_{-k}^+ + \bar{c}_{-k} \bar{c}_k) +$$
$$+ u_k^2 \bar{c}_{-k} c_{-k}^+ \Big) +$$
$$+ J_2 \sin k \Big(u_k^2 \bar{c}_k \bar{c}_{-k} + u_k v_k (\bar{c}_{-k}^+ \bar{c}_{-k} - \bar{c}_k \bar{c}_k^+) -$$
$$- v_k^2 \bar{c}_{-k}^+ \bar{c}_k^+ \Big) + h.c.$$

Now let us look for u_k, v_k such that the coefficient of $\bar{c}_{-k} \bar{c}_k$ is $= 0$. Then

$$2 u_k v_k (J_1 \cos k - \mu) + (-u_k^2 + v_k^2) J_2 \sin k = 0.$$

This gives us

$$\frac{v_k}{u_k} = \frac{\mu - J_1 \cos k \pm \sqrt{(J_1 \cos k - \mu)^2 + J_2^2 \sin^2 k}}{J_2 \sin k}.$$

Solving this with $u_k^2 + v_k^2 = 1$ and substituting the result into $\hat{H}$ gives.

$$\hat{H} = \text{const} + \sum_k \left[2(J_1 \cos k - \mu)(u_k^2 - v_k^2) + 4 J_2 \sin k \, u_k v_k \right] \times$$

$$\times c_k^+ c_k =$$

$$= \text{const} + \sum_k \varepsilon_k c_k^+ c_k,$$

where

$$\varepsilon_k = 2\sqrt{(J_1 \cos k - \mu)^2 + J_2^2 \sin^2 k}.$$

§ Absence of dispersion at $J_1 = J_2$, $\mu = 0$ means that excitations are localized within one or a few sites of the chain.

Solution 2: Let us consider the Fourier transformed form of the Hamiltonian:

$$\hat{H} = \sum_k \left[2(J_1 \cos k - \mu) c_k^+ c_k - iJ_2 \sin k\, c_k c_{-k} + iJ_2 \sin k\, c_{-k}^+ c_k^+ \right]$$

Then, one can compute a commutator $[\hat{H}, c_k]$ using the following identities:

$$[c^+c, c] = c^+cc - cc^+c = -c$$

$$[cc^+, c^+] = -c^+.$$

One will thus obtain

$$[\hat{H}, c_k] = -2(J_1 \cos k - \mu) c_k + 2J_2 i \sin k\, c_{-k}^+.$$

Let us remember that the physical meaning of the commutator of a given operator with $\hat{H}$ is

$$i\hbar \dot{A} = [A, \hat{H}]$$ -> is the "speed of change" of A.

Now let us consider canonical transformation

$$c_k = u_k b_k + V_k b_{-k}^+ \quad , \quad c_{-k}^+ = u_k^* b_{-k}^+ - V_k^* b_k ,$$

and let us assume that parameters u_k, v_k are chosen in such a way that the Hamiltonian is diagonal in b_k, b_k^+:

$$\hat{H} = E_0 + \sum_k{}' \varepsilon_k b_k^+ b_k .$$

Now let us compute the following commutators

$$[\hat{H}, \hat{b}_k] = -\varepsilon_k b_k , \quad [\hat{H}, b_k^+] = \varepsilon_k b_k^+ .$$

Using these expressions, one can rewrite the commutator $[\hat{H}, c_k]$ as follows:

$$\left[\hat{H}, c_k\right] = \left[E_0 + \sum_k \varepsilon_k b_k^+ b_k , \; u_k b_k + v_k b_{-k}^+\right] =$$

$$= -u_k \varepsilon_k b_k + v_k \varepsilon_k b_{-k}^+$$

On the other hand, the RHS of the expression of $[\hat{H}, c_k]$ on page 15 will acquire the form:

$$[H, c_k] = -u_k \varepsilon_k b_k + v_k \varepsilon_k b_{-k}^+ = -2\left(J_1 \cos k - \mu\right)\left(u_k b_k + v_k b_{-k}^+\right)$$

$$+ 2 J_2 i \sin k \left(u_k^* b_{-k}^+ - v_k^* b_k\right) .$$

From here one obtains equations for $\mathcal{E}_k, u_k, v_k$:

$$u_k \mathcal{E}_k = 2\left(J_1 \cos k - \mu\right) u_k + 2 J_2\, i \sin k\, v_k^* \qquad (1)$$

$$v_k \mathcal{E}_k = -2\left(J_1 \cos k - \mu\right) v_k + 2 J_2\, i \sin k\, u_k^* \qquad (2)$$

Since these equations are homogeneous, one can solve (2) for v_k:

$$v_k = \frac{2 J_2\, i \sin k\, v_k^*}{\mathcal{E}_k + 2\left(J_1 \cos k - \mu\right)},$$

and then substitute the solution into (1) $\Longrightarrow$

$$\Longrightarrow \quad \mathcal{E}_k^2 - 4\left(J_1 \cos k\right)^2 = 4 J_1^2 \sin^2 k.$$

From here: $\mathcal{E}_k = 2\sqrt{\left(J_1 \cos k - \mu\right)^2 + J_2^2 \sin^2 k}$.

Notice that we choose + sign, as the ~~one~~ excitation energy above the ground state is always positive.

Parameters u_k, v_k can be found from the above equation noting that $u_k^2 + v_k^2 = 1$ $\rightarrow$ normalization condition.

Fradkin Chapter 5 | Reading: Coleman, Chapter 4

Jordan-Wigner transformation

A non-interacting gas of fermions is still highly correlated due to the exclusion principle. This is ~~ref~~ exploited in the Jordan-Wigner representation of spins.

A classical spin is a vector - a good representation for quantum spins with large S. At small S, ~~ene~~ it is not good. What happens at $S = 1/2$?

$\boxed{\text{First (naive!) attempt}}$

$$|S\rangle = \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases} \Rightarrow$$ these states can be thought of as empty and singly occupied fermion states:

$$|\uparrow\rangle \equiv f^{+}|0\rangle, \qquad |\downarrow\rangle \equiv |0\rangle$$

An explicit representation of the spin-raising and spin-lowering operators is then

$$S^{+} = f^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S^{-} = f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

For z-component thus one has

$$S_z = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right) \equiv f^{+}f - 1/2$$

Reconstructing transverse spin operators we see:

$$\left.\begin{aligned} S_x &= \frac{1}{2}(S^{+}+S^{-}) = \frac{1}{2}(f^{+}+f) \\ S_y &= \frac{1}{2i}(S^{+}-S^{-}) = \frac{1}{2i}(f^{+}-f) \end{aligned}\right\}$$

<u>majorana fermions:</u>

$\xi = \frac{1}{2}(f^{+}+f) \Rightarrow$

$\Rightarrow \xi^{+} \equiv \xi$. Similarly

$\bar{\xi} = \frac{1}{2i}(f^{+}-f) \Rightarrow \bar{\xi}^{+} \equiv \bar{\xi}$.

The explicit matrix representation of these operators makes it clear that

$$[S_a, S_b] = i\,\varepsilon_{abc}\, S_c$$ – the correct algebra SU(2

However, due to "fermionic" nature of f-s

$$\{S_a, S_b\} = \frac{1}{4}\{\sigma_a, \sigma_b\} = \frac{1}{2}\delta_{ab}, \qquad S_a = \frac{1}{2}\sigma_a,\ a = 1,2,3$$

where $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ –

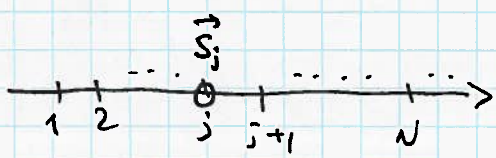
are Pauli matrices with properties $\mathrm{Tr}\,\sigma_i = 0$ and $\det\sigma_i = -$

Full fermionization in 1D:

~~Th~~ If there is more then one spin in the system; we know that

* independent spin operators commut
* independent fermions anticommute.

Jordan-Wigner transform: to fix this in 1D, let us attach a string to the fermion at site j



$$\hat{S}_j^+ = \hat{f}_j^+ e^{i\hat{\phi}_j},$$ where

the phase operator $\hat{\phi}_j$

containes the sum over all fermion occupancies of sites to the left of j:

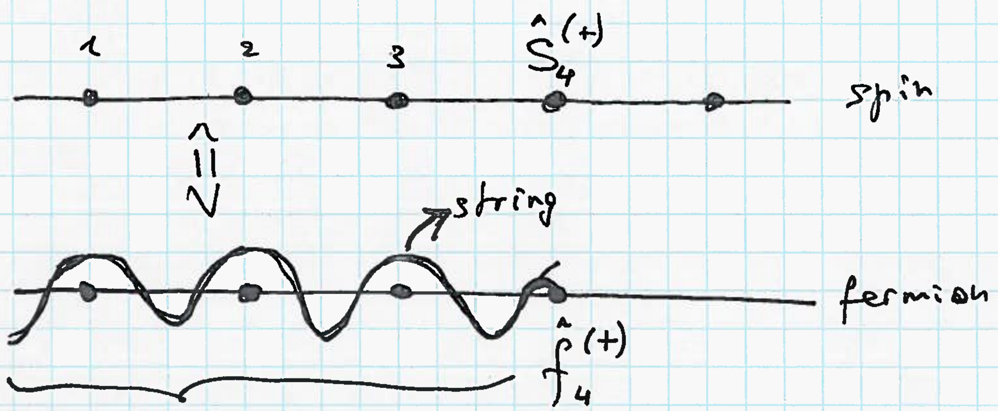
$$\phi_j = \pi \sum_{\ell<j} \hat{n}_j .$$ $e^{i\phi_j}$ – is called a string operator.

The complete transform is then

$$\tilde{S}_j^z = \hat{f}_j^+ \hat{f}_j - \frac{1}{2}$$
$$\tilde{S}_j^+ = \hat{f}_j^+ \, e^{i\pi \sum_{\ell<j} \hat{n}_j}$$
$$\tilde{S}_j^- = \hat{f}_j \, e^{-i\pi \sum_{\ell<j} \hat{n}_j}$$

where $e^{i\pi n_j} = e^{-i\pi n_j}$ is a Hermitian operator.

<u>Important property</u>: the string operator $e^{i\hat{\phi}_j}$ anticommutes with any fermion operator to the left of its free end



string $\equiv e^{i\pi(n_1+n_2+n_3)}$

<u>proof</u>: a) let us see that $e^{i\pi n_j}$ anticommutes with f_j:

$$\{ e^{i\pi n_j}, f_j \} = e^{i\pi n_j} f_j + f_j e^{i\pi n_j}$$

$$e^{i\pi \hat{n}} = 1 + i\pi n + \frac{1}{2}(i\pi n)^2 + \dots = 1 + i\pi n + \frac{n}{2}(i\pi)^2 + \dots \frac{n}{\ell!}(i\pi)^\ell$$

$$= 1 + n\left(\sum_{\ell=0}^{\infty} \frac{(i\pi)^\ell}{\ell!} - 1 \right) = 1 + n\left(e^{i\pi} - 1\right) = 1 - 2n$$

$$\Rightarrow \{ e^{i\pi n_j}, f_j \} = (1-2n_j) f_j + f_j (1-2n_j) =$$

$$= \underbrace{-2 f_j^+ f_j f_j}_{=0} + \underbrace{2 f_j f_j^+ f_j}_{2f_j} + 2 f_j = 0.$$

and similarly, $\{e^{i\pi n_j}, f_j^+\} = 0$.

b) Now, the phase $e^{i\pi\hat{n}_\ell}$ at any $\underline{\ell \neq j}$ commutes with f_j and f_j^+ => the string operator $e^{i\hat{\phi}_j}$ anticommutes with all fermions at all sites ℓ to the left of j:

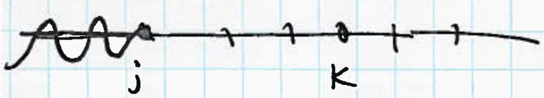
$$\{e^{i\phi_j}, f_\ell^{(+)}\} = 0 \quad , \quad \ell < j$$

while it commutes with fermions at all other sites

$$[e^{i\hat{\phi}_j}, f_\ell^{(+)}] = 0 \quad , \quad \ell \geq j.$$

Let us now verify that the transverse spin operators satisfy the correct commutation algebra.

suppose $\underline{j < k}$ => $e^{i\phi_j}$ commutes with fermions at sites j and k =>



$$[S_j^\pm, S_k^\pm] = [f_j^{(+)} e^{i\phi_j}, f_k^{(+)} e^{i\phi_k}] =$$

$$= e^{i\phi_j}[f_j^{(+)}, f_k^{(+)} e^{i\phi_k}]$$

Here, $f_j^{(+)}$ anticommutes with both, $f_k^{(+)}$ and $e^{i\hat{\phi}_k}$ => it commutes with their product $f_k^{(+)} e^{i\phi_k}$

$$\Rightarrow [S_j^{(\pm)}, S_k^{(\pm)}] = e^{i\phi_j}\underbrace{[f_j^{(+)}, f_k^{(+)} e^{i\phi_k}]}_{=0} = 0$$

Example: 1D Heisenberg chain (XXZ)

$$H = -J \sum \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y \right] - J_z \sum_j S_j^z S_{j+1}^z .$$

Can be rewritten as:

$$H = -\frac{J}{2} \sum \left[S_{j+1}^+ S_j^- + \text{H.c.} \right] - J_z \sum_j S_j^z S_{j+1}^z .$$

Fermionization yields:

$$\frac{J}{2} \sum_j S_{j+1}^+ S_j^- = \frac{J}{2} \sum_j f_{j+1}^+ e^{i\pi n_j} f_j = \underbrace{\frac{J}{2} \sum_j f_{j+1}^+ f_j}_{\text{hopping of fermions.}} ,$$

For the z-component, one obtains:

$$-J_z \sum_j S_{j+1}^z S_j^z = -J_z \sum_j \left(n_{j+1} - \frac{1}{2}\right)\left(n_j - \frac{1}{2}\right),$$

and thus ferromagnetic spin-interaction means that spin fermions attract each other.

Fermionized Hamiltonian thus reads:

$$H = -\frac{J}{2} \sum_j \left(f_{j+1}^+ f_j + f_j^+ f_{j+1} \right) + J_z \sum_j n_j - J_z \sum_j n_j n_{j+1}$$

While fermionization of XYZ model:

$$\hat{H}_{XYZ} = -\sum_{i=-\infty}^{\infty} \left(J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z \right) - \mu S_i^z$$

gives:

$$\hat{H}_{XYZ} = \sum_{i=-\infty}^{\infty} \left(J_1 c_i^+ c_{i+1} + J_2 c_i c_{i+1} + h.c. \right) - \mu \left(n_i - \tfrac{1}{2}\right) + J_z \left(n_i - \tfrac{1}{2}\right)\left(n_{i+1} - \tfrac{1}{2}\right) , \quad J_{1,2} = \frac{J_x \pm J_y}{4} .$$