Block summation of Feynman diagrams

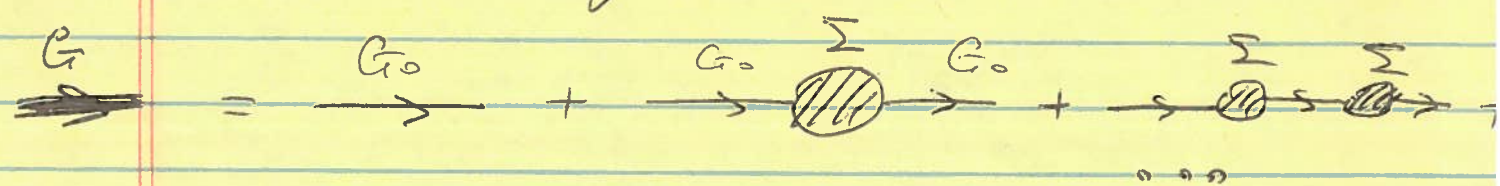
In diagrammatic approach, one is able not only to calculate corrections in a given order of perturbation theory, but also to perform block summation of certain infinite series of diagrams.

Within such an approach, sometimes, one just needs to calculate a certain number of diagrams (blocks), and then perform an infinite series summation using these results.

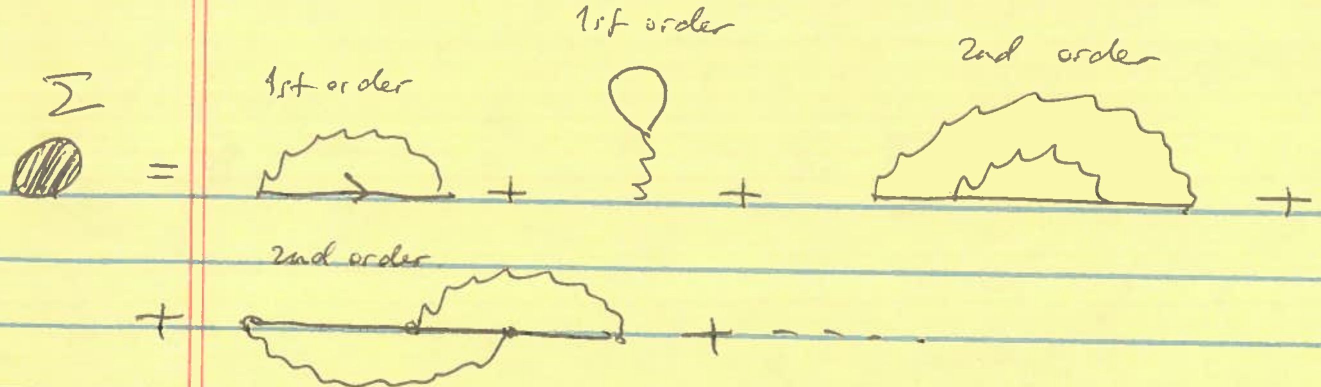
~~The~~ Single-particle Green's function:

Consider an arbitrary diagram that contributes to $G(\xi\vec{p})$. The FD is either reducible or irreducible.

Any diagram contributing to the Green's function ~~can be rep~~ can be schematically represented as one of the diagrams below:



where Σ - objects are irreducible diagrams called Self-energy diagram.



Performing ~~sum~~ diagrammatic summation for Green's function, one obtains

$$G = G_0 + G_0 \Sigma G_0 + G_0 \Sigma G_0 \Sigma G_0 + \ldots =$$

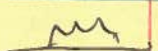
$$= G_0 + G_0 \Sigma G$$ - is called Dyson equation. (real space)

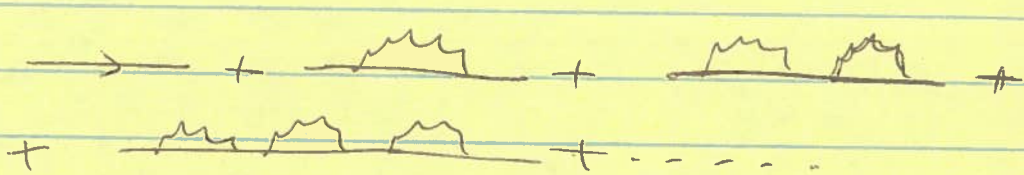
The solution of Dyson's equation can be cast as

$$G^{-1}(\varepsilon, \vec{p}) = G_0^{-1}(\varepsilon, \vec{p}) - \Sigma(\varepsilon, \vec{p}).$$

Dyson's equation establishes a connection between $G(\varepsilon, \vec{p})$ and $\Sigma(\varepsilon, \vec{p})$, ~~and~~ i.e., the sums of all reducible and irreducible diagrams.

Generically, $\Sigma(\varepsilon, \vec{p})$ is <u>hard</u> to calculate exactly but calculating the first-order ~~co~~ diagram

$\Sigma_1 =$ , one may immediately perform infinite summation for $G(\varepsilon,\vec{p})$:

$G^{(1)} =$ 

Then, one may use $G^{(1)}(\varepsilon,\vec{p})$ instead of $G_0(\varepsilon,\vec{p})$ to go beyond this approximation giving $G^{(1)}$

Poles of the Green's function: quasiparticle spectrum

For noninteracting Fermi-particles we have that

$$G_0(\varepsilon,\vec{p}) = \frac{1}{\varepsilon - \xi(\vec{p}) + i\delta(\vec{p})}, \quad \xi(\vec{p}) = \frac{\vec{p}^{\,2}}{2m} - E_F -$$

– dispersion relation.

$i\delta(\vec{p}) = i\,0\cdot \mathrm{sign}(|\vec{p}| - p_F)$. All states with $|p| < p_F$ are filled and others are empty.

~~At $|\vec{p}| >$~~
* Excitations with $|\vec{p}| > p_F$ are "particles"
* Excitations with $|\vec{p}| < p_F$ are "holes".

The reason is the following: In Heisenberg's representation, where the "heat bath"'s energy is subtracted:

$$H = \hat{H}_0 - E_F \cdot N = \sum_\sigma \xi_{\vec{p}}\, \hat{c}^+_{p\sigma} \hat{c}_{p\sigma}$$

is the Hamiltonian in Heisenberg representation,

Then ~~we take~~ $c^+_{p\sigma}(t) = e^{i\varepsilon_p t} c_{p\sigma}$

$$c_{p\sigma}(t) = e^{-i\varepsilon_p t} c_{\vec{p}\sigma},$$

$\boxed{\varepsilon_p \equiv \xi_p \text{ below.}}$

and the ground state for a fluid of fermions is

$$|\Psi\rangle = \prod_\sigma \prod_{|p|<p_F} c^+_{\vec{p}\sigma} |0\rangle.$$

Then

$$\langle\Psi| c_{\vec{p}\sigma}(t)\, c^+_{\vec{p}'\sigma'}(t')|\Psi\rangle = \delta_{\sigma\sigma'}\delta_{\vec{p}\vec{p}'} e^{-\varepsilon_p(t-t')} \times$$

$$\times \langle\Psi| c_{p\sigma} c^+_{p\sigma} |\Psi\rangle = \delta_{\sigma\sigma'}\delta_{\vec{p}\vec{p}'}(1-n_{\vec{p}})e^{-i\varepsilon_p(t-t')}$$

where $n_{\vec{p}} = \theta(p_F - |\vec{p}|)$.

Therefore, $G(\vec{p}, t) = -i\left[(1-n_{\vec{p}})\theta(t) - n_{\vec{p}}\theta(-t)\right]e^{-i\varepsilon_p t}$

$$= \begin{cases} -i\,\theta_{|\vec{p}|-|\vec{p}_F|}\, e^{-i\varepsilon_{\vec{p}} t} & (t>0) \text{ particles} \\ i\,\theta_{|\vec{p}_F|-|\vec{p}|}\, e^{-i\varepsilon_{\vec{p}} t} & (t<0) \text{ holes} \Leftrightarrow \end{cases}$$

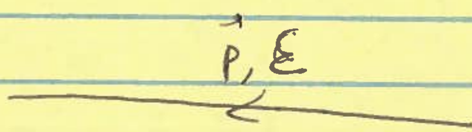
particles moving backwards in time

Fourier tranform yields:

$$G(\vec{p},\varepsilon) = -i\int_{-\infty}^{\infty} dt\, e^{i(\varepsilon-\varepsilon_{\vec{p}})t}\, e^{-|t|\cdot\delta}\left[\theta_{p-p_F}\,\theta(t) - \theta_{p_F-p}\,\theta(-t)\right]$$

$$= -i\left[\frac{\theta_{p-p_F}}{\delta - i(\varepsilon-\varepsilon_{\vec{p}})} - \frac{\theta_{p_F-p}}{\delta + i(\varepsilon-\varepsilon_{\vec{p}})}\right] = \frac{1}{\varepsilon - \varepsilon_p + i\delta_p}$$

where $\delta_p = \delta \cdot \mathrm{Sign}(p - p_F)$.

$G(\vec{p},\varepsilon) =$ 

Now the exact Green's function

$$G(\varepsilon,\vec{p}) = -i\int \langle T\, \psi(\vec{r},t)\,\psi^{+}(0,0)\rangle\, e^{-i\varepsilon t - i\vec{p}\vec{r}}\, d^3r\,dt,$$

where the interaction are taken into account, often ~~descr~~ (not always) ~~describes~~ distinguishes the quasiparticles as the poles ~~of~~ in complex plain of ε:

$$G(\varepsilon,\vec{p}) = \frac{Z}{\varepsilon - \xi(\vec{p}) + i\gamma(\vec{p})} + G_{reg}(\varepsilon,\vec{p}).$$

here Z is called a Z-factor, G_{reg} is a regular function near $\mathcal{E} = \xi(p) - i\gamma(\vec{p})$.

The Quasiparticle spectrum corresponds to the pole in and is given by the function $\xi(\vec{p})$.

Damping term $\gamma(\vec{p}) = \frac{1}{2\tau_p}$, where τ_p is the lifetime of quasiparticles. Dispersion relation $\mathcal{E} = \xi(\vec{p})$ defines the so-called mass shell of quasiparticles in 4D space of $(\mathcal{E}, \vec{p})$.

Generally the notion of quasiparticles is well defined if their lifetime is sufficiently long

$$\gamma(\vec{p}) << \xi(\vec{p}).$$

In the opposite situation, there are damped excitations.

From the solution of Dyson's equation it is clear that the self-energy term $\Sigma(\mathcal{E}, \vec{p})$ defines the quasiparticle spectrum. Investigating Im and Re parts of the solution of Dyson's equation, we see that

$$\xi(\vec{p}) = \xi_0(\vec{p}) + Re\,\Sigma\big(\xi(\vec{p}), \vec{p}\big)$$

$$\gamma(\vec{p}) = Im\,\Sigma\big(\xi(\vec{p}), \vec{p}\big).$$

$\Rightarrow$ Re Σ renormalizes the dispersion relation. It is responsible e.g., for interaction renormalization of the particle mass. $\Leftrightarrow$ effective mass.

$\Rightarrow$ Im Σ characterizes damping of quasiparticles.

$\tau_p = \frac{1}{2\delta(\vec{p})}$ - is the quasiparticle lifetime with momentum $\vec{p}$.

Because of causality, Re Σ and Im Σ are connected with each other via Kramers-Kronig relations.

Two-body Green's function

The ~~single~~ Green's function $G(\varepsilon, \vec{p})$ describes propagation of a single quasiparticle, but it does not provide any information about <u>bound states</u> of two particles. To describe latter, one introduces a two-particle Green's function

$$K_{ab} = \pm \langle T\, C_a(x_1)\, C_b(x_2)\, C_a^{+}(x_3)\, C_b^{+}(x_4) \rangle,$$

where + corresponds to bosons, − corresponds to fermions, $x = (t; \vec{r})$. Indices a, b distinguishes between particles.

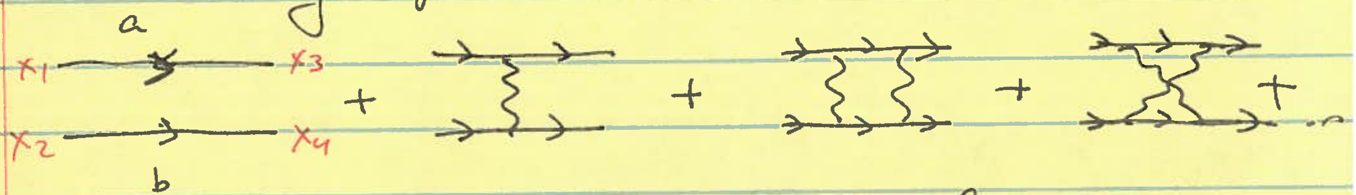
For noninteracting particles

$$K_{ab}(x_1, x_2, x_3, x_4) = G_a(x_1 - x_3)\, G_b(x_2 - x_4)$$

(If the particles are identical, then depending on statistics of these particles, one has to add or subtract the product of the Green's functions with switched arguments).

Perturbation theory for K_{ab}:

$K_{ab} =$



where it is assumed that the Green's functions in these diagrams are exact (bold Green's functions).

It is convenient to introduce the vertex part of the diagram by writing

$$K_{ab}(x_1, x_2, x_3, x_4) = G_a(x_1 - x_3)\, G_b(x_2 - x_4) + i \int G_a(x_1 - x_1')\, G_a(x_3' - x_3)$$
$$\times\, G_b(x_2 - x_2')\, G_b(x_4' - x_4)\, \Gamma_{ab}(x_1', x_2'; x_3', x_4') \cdot d^4x_1'\, d^4x_2'\, d^4x_3'\, d^4x_4',$$

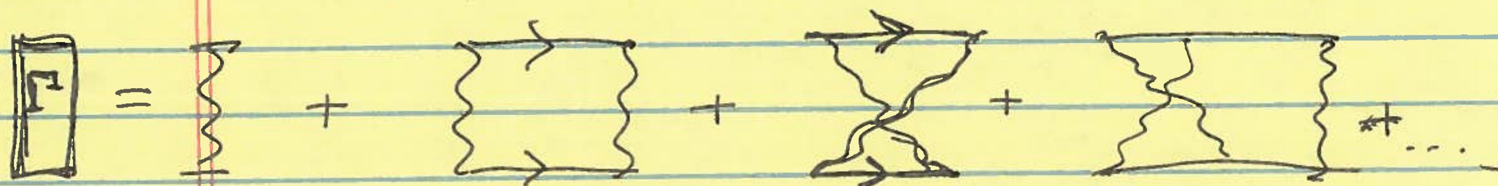
where $\Gamma_{ab}(x_1', x_2'; x_3', x_4')$ describes the interaction of particles (sometimes called 2-particle scattering amplitude)

In momentum representation Γ_{ab} reads

$$\Gamma_{ab}(p_1, p_2, p_3, p_4) = \int \Gamma_{ab}(x_1, x_2; x_3, x_4)\, e^{-i(p_1 x_1 + p_2 x_2 + p_3 x_3 + p_4 x_4)}$$
$$\times d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4,$$

where $p_i = (\varepsilon_i, \vec{p}_i)$ – are 4D momenta. Diagrammatically

Γ_{ab} can be represented as



In $\Gamma_{ab}(p_1, p_2, p_3, p_4)$ the sum of 4D momenta is always conserved:

$$p_1 + p_2 = p_3 + p_4.$$

In these diagrams: $\longrightarrow \equiv G(p)$

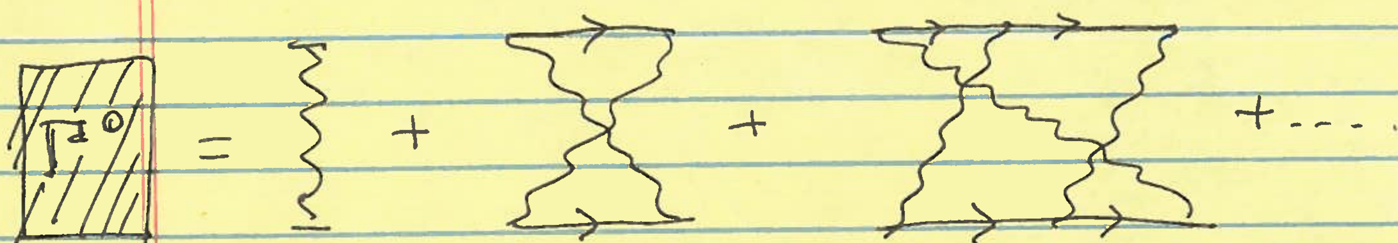
$\sim\!\sim\!\sim \equiv U(p)$

and there is integration over each internal momentum.

One can perform nonperturbative summation of infinite # of series of diagrams.

A diagram is called 2-particle irreducible, if it does not become disconnected after cutting *any* two lines corresponding to Green's functions.

The sum of all 2-particle irreducible diagrams defines the so called irreducible vertex part Γ^0:



Given infinite series for Γ^0, one can now write *any* diagram for Γ_{ab} as a series of irreducible diagrams, which are connected to each other via pairs of Green's functions $G(p)$.

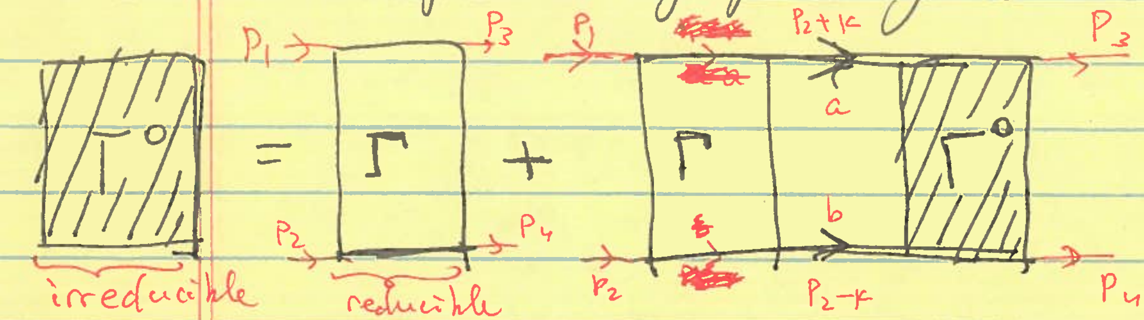
Thus one obtains Bethe-Salpeter equation for vertex function Γ_{ab}:

$$\Gamma_{ab}(p_i) = \Gamma^0_{ab}(p_i) + i\int \Gamma^0_{ab}(p_i')\, G_a^{(1)}(p_1+k)\, G_b^{(2)}(p_2-k)\, \Gamma_{ab}(p_i'') \times \frac{d^4k}{(2\pi)^4}$$

where $p_i = \{p_1, p_2, p_3, p_4\}$, $p_i' = \{p_1, p_2, p_1+k, p_2-k\}$,

$$p_i'' = \{p_1+k, p_2-k, p_3, p_4\}$$

which can be represented graphically as



Like a single-particle Green's function, also 2-particle scattering amplitude (vertex function) can have poles. These poles correspond to bound states of pairs of particles.

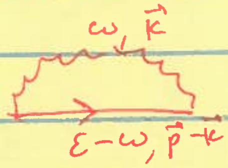
<u>Examples:</u> polaron and week coupling approximation.

Electrons in the conduction band of the semiconductor represent a dilute gas. At small electron concentrations, interparticle interactions can be neglected.

At the same time, an individual electron, moving in a cristal lattice, polarizes the medium surrounding it and induces lattice deformation that follows it. => ~~polar~~ This electron dressed with lattice deformation is called <u>polaron</u>.

Polaron dispersion: $\varepsilon(\vec{p}) = \varepsilon_0 + \frac{\vec{p}^{\,2}}{2m^*}$, $m^* \neq m$_elect mass.

Let us find the self-energy of the polaron, $\Sigma(\varepsilon, \vec{p})$ in lowest order perturbation theory in electron-phonon interactions.

$\Sigma =$ 

, where $G_0(\varepsilon, \vec{p}) = \dfrac{1}{\varepsilon - p^2/2m + i\delta}$

$\delta > 0 \Rightarrow$ there are no electrons in conduction band.

 $= D_0(\omega, \vec{k}) = \dfrac{c^2 \vec{k}^2}{\omega^2 - c^2\vec{k}^2 + i\delta}$ - acoustic phonon propagator, c-speed of sound.

$$\Sigma(\varepsilon, \vec{p}) = i g^2 \int G_0(\varepsilon - \omega)\, D_0(\omega, \vec{k}) \frac{d^3k}{(2\pi)^3} \frac{d\omega}{2\pi}$$

* Integration over ω can be performed by pole integration in lower half-plane:

$$\int_{-\infty}^{\infty} \frac{1}{\tilde{\varepsilon} - \omega + i\delta} \underbrace{\frac{c^2k^2}{\omega^2 - c^2k^2 + i\delta}}_{\uparrow\ \text{poles}} \frac{d\omega}{2\pi} = \frac{i}{2} \frac{ck}{ck - \tilde{\varepsilon} - i\delta},$$

where $\tilde{\varepsilon} = \varepsilon - (\vec{p} - \vec{k})^2/2m$, $k = |\vec{k}|$. This gives

$$\Sigma(\varepsilon,\vec{p}) = \frac{g^2}{\alpha}\int \frac{ck}{\varepsilon - ck - (\vec{p}-\vec{k})^2/2m + i\delta}\cdot\frac{d^3k}{(2\pi)^3}.$$

Let $x = \cos(\vec{p}\,\hat{}\,\vec{k})$. $\Rightarrow d^3k = 2\pi k^2\,dk\,dx$.

Introduce $q^2 = |\vec{p}-\vec{k}|^2 = p^2 + k^2 - 2pkx \Rightarrow$

$$\Rightarrow q\,dq = -pk\,dx \Rightarrow dx = -\frac{q}{pk}dq$$

Then $$\int f(|\vec{k}|, |\vec{p}-\vec{k}|)\frac{d^3k}{(2\pi)^3} = \frac{1}{(2\pi)^2}\cdot\frac{1}{p}\int k\,dk\int_{|p-k|}^{|p+k|} f(k,q)\,q\,dq$$

where we integrate over $|\vec{k}| = k$ and $q = |\vec{p}-\vec{k}|$.

This gives: $$\Sigma = \frac{g^2}{2(2\pi)^2 p}\cdot\int_0^{k_D} k\,dk\int_{|\vec{p}-\vec{k}|}^{|\vec{p}+\vec{k}|}\frac{ck\,q\,dq}{\varepsilon - q^2/2m - ck + i\delta}$$

Integration over $y = q^2$ yields:

$$\Sigma = \frac{g^2 mc}{8\pi^2 p}\int_0^{k_D}\ln\left|\frac{\varepsilon - (p-k)^2/2m - ck}{\varepsilon - (p+k)^2/2m - ck}\right| k^2\,dk -$$

$$- i\,\frac{g^2 mc}{8\pi p}\int_0^{k_D} k^2\,dk\int_{(p-k)^2}^{(p+k)^2}\delta\big(y - 2m(\varepsilon - ck)\big)\,dy$$

At $p < mc$ the self-energy part is real $\Rightarrow$ polaron is stable!

Effects associated with decays at $p > mc$ can be studied separately.

Near the mass-shell $\varepsilon = \frac{p^2}{2m}$ and at small p we can expand the exact Σ over small $\Delta = \left(\varepsilon - \frac{p^2}{2m}\right)$:

$$\Sigma = \frac{g^2 mc}{8\pi^2 p} \int_0^{k_D} \ln \left| \frac{k^2/2m + (c-v)k - \Delta}{k^2/2m + (c+v)k - \Delta} \right| k^2 dk =$$

$$= \frac{g^2 mc}{8\pi^2 p} \int_0^{k_D} \left(-\frac{2vk}{ck + k^2/2m} - \frac{4\Delta vk}{2(ck + k^2/2m)^2} - \frac{2v^3k^3}{3(ck + k^2/2m)^3} + \ldots \right) k^2 dk$$

These three terms give us: $\Sigma = \varepsilon_0 - \alpha_1 \Delta - \alpha_2 \frac{p^2}{2m}$,

where integration over k ~~can be~~ is easily performed since $c \ll k_D/m$ and (ck)-terms in denominators is negligible w.r.t. $k^2/2m$: $ck \ll \frac{k^2}{2m}$ almost everywhere in integration bounds.

So we obtain:

$$\mathcal{E}_0 = -\frac{g^2 c}{4\pi^2}\int_0^{k_D}\frac{k^3\,dk}{ck + k^2/2m} = -\frac{g^2 c\, k_D^2\, m}{4\pi^2}$$

and within logarithmic accuracy:

$$\alpha_1 = \frac{g^2 m^2 c}{\pi^2}\int_0^{k_D}\frac{k^3\,dk}{(k^2+2mck)^2} = \frac{g^2 m^2 c}{\pi^2}\ln\left(\frac{k_D}{mc}\right).$$

$$\alpha_2 = \frac{2g^2 m^3 c}{3\pi^2}\cdot\frac{2}{m}\int_0^{k_D}\frac{k^5\,dk}{(k^2+2mck)^3} = \frac{4g^2 m^2 c}{3\pi^2}\ln\left(\frac{k_D}{mc}\right)$$

$$= \frac{4}{3}\alpha_1.$$

Dispersion relation $G_0^{-1} - \Sigma_1 = 0$ → on mass-shell! gives mass renormalization:

$$\boxed{\frac{m_*}{m} = 1 + \alpha_2 > 1} \Rightarrow$$

⇒ electron is dressed (and not undressed) as the correction to the mass is positive.