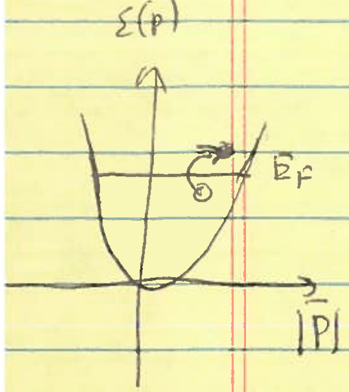
Fermi Gas



Near Fermi level, the gas has an important symmetry: particle-hole symmetry which is there when the energy of particle-hole excitations is $\ll E_F$.

This symmetry is approximate, and is broken at energies away from E_F.

Kubo formula:

Consider the dynamical susceptibility that describes a linear responce of the system w.r.t. an external time-dependent field: susceptibility of ~~A~~ quantity A with respect to B is given by Kubo formula

$$\chi(\omega) = \frac{i}{\hbar}\int_0^\infty \langle [\hat{A}(t), \hat{B}(0)] \rangle e^{i\omega t}\, dt.$$

To calculate the expectation value ~~of~~ for non-interacting particles, let us write operators $\hat{A}$, $\hat{B}$ in second-quantized form:

$$\hat{A}(t) = \sum_{m,k} A_{mk}\psi_m^+\psi_k e^{-i(E_k-E_m)t}, \quad \hat{B}(t) = \sum_{m,k}\psi_m^+\psi_k B_{mk} e^{-i(E_k-E_m)}.$$

Substituting these expressions into Kubo formula, one obtains

$$\chi(\omega) = \sum_{m,k} A_{mk} B_{km} \frac{n(E_m) - n(E_k)}{E_k - E_m - \omega - i\delta},$$

where $n(E)$ is the distribution function

$$n_{F/B}(E) = \frac{1}{e^{(E-\mu)/k_B T} \pm 1}.$$

If martix elements are known, and the wavefunctions are known $\Rightarrow$ it is straightforward to compute χ

If the eigenfunctions aren't known, one may rewrite $\chi(\omega)$ in terms of Green's functions.

For this purpose, consider the energy-dependent denominator of $\chi(E)$:

$$\frac{1}{E_k - E_m - \omega - i\delta} = \frac{1}{2\pi i} \int \frac{d\varepsilon}{(\varepsilon + \omega - E_k - i\delta)(\varepsilon - E_m + i\delta)} =$$

$$= \frac{1}{2\pi i} \int G_k^A(\varepsilon + \omega)\, G_m^R(\varepsilon)\, d\varepsilon,$$

where $G_k^{A/R}$ are Green's functions in diagonal representation, From here, one obtains that

$$\varkappa(\omega) = \frac{1}{2\pi i} \int \mathrm{Tr}\left(\left[\hat{G}^{A}(\varepsilon+\omega)\,\hat{B} ; \hat{G}^{R}(\varepsilon)\,\hat{A} \right] \hat{\rho} \right) d\varepsilon,$$

where $\hat{\rho}$ is the density matrix of the system (in diagonal representation $\hat{\rho}_{mk} = n(E_m)\,\delta_{mk}$).

This expression works in $\forall$ basis $\Rightarrow$ can be used if we do not know eigenfunctions of the system.

Let us first calculate electronic Green's function in real space: $G(\varepsilon, \vec{r}_1 - \vec{r}_2)$.

As a reminder, causal Green's function is defined as

$$G^c_{\alpha\beta}(x, x') = -i \langle T \psi_\alpha(x) \psi^+_\beta(x') \rangle,$$

$x = (\vec{r}, t)$. Advanced and retarded Green's functions are connected with the causal one as follows:

$$G^c(t,t') = \begin{cases} G^R(t,t'); & t > t' \\ G^A(t,t'), & t < t'. \end{cases}$$

In energy representation, $G^R(\varepsilon)$, $G^A(\varepsilon)$ define regular (with no poles) parts of $G^c(\varepsilon)$ in upper and lower half-planes of ω correspondingly

$$G^{R/A}(\varepsilon, \vec{p}) = \frac{1}{\varepsilon - \xi(\vec{p}) \pm i\delta}, \quad \text{and}$$

the causal G^c is thus

$$G^c(\omega, \vec{p}) = [1 - n(\vec{p})] \cdot G^R(\xi \vec{p}) + n(\vec{p})\, G^A(\xi, \vec{p}) =$$

$$= \underbrace{\frac{1 - n(\vec{p})}{\varepsilon - \xi(\vec{p}) + i\delta}}_{\text{particles}} + \underbrace{\frac{n(\vec{p})}{\varepsilon - \xi(\vec{p}) - i\delta}}_{\text{holes}},$$

$$n(\vec{p}) = \begin{cases} 1, & |\vec{p}| < p_F \\ 0, & |\vec{p}| > p_F \end{cases}$$ – is Fermi-distribution function.

Then the causal Green's function in coordinate-frequency representation reads:

$$G(\varepsilon, \vec{r}) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} p^2 \frac{dp}{2\pi} \sin\theta \frac{d\theta}{2\pi}\frac{d\varphi}{2\pi} \frac{e^{ipr\cos\theta}}{\varepsilon - \xi(\vec{p}) + i\delta\, \mathrm{Sign}(\varepsilon)} =$$

$$= \int_0^{\pi}\int_0^{\infty} \frac{p^2 dp \sin\theta\, d\theta}{(2\pi)^2} \frac{e^{ipr\cos\theta}}{\varepsilon - \xi(\vec{p}) + i\delta\, \mathrm{Sign}(\varepsilon)},$$

where $\xi(\vec{p}) = \frac{p^2}{2m} - E_F$

Using the fact that

$$\int_0^{\pi} e^{ix\cos\theta} \sin\theta\, d\theta = \frac{2\sin x}{x}$$ Since

$$= -\int_0^{\pi} e^{ix\cos\theta}\, d\cos\theta = -\int_{u=-1}^{1} du\, e^{\underbrace{ixu}_{y}} =$$

$$= \frac{1}{x}\int_{-x}^{x} e^{iy} dy = \frac{1}{ix} e^{iy}\Big|_{-x}^{x} = \frac{2\sin x}{x}$$

$$G(\varepsilon, \vec{r}) = \frac{1}{2\pi^2 r} \int_0^{\infty} \frac{p \sin(pr)\, dp}{\varepsilon - \xi(\vec{p}) + i\delta\, \mathrm{Sign}\,\varepsilon} =$$

$$= \frac{1}{4\pi^2 r} \int_{-\infty}^{\infty} \frac{p \sin pr \, dp}{\varepsilon - p^2/2m + E_F + i\delta \, \mathrm{Sign}\,\varepsilon} =$$

$$= \frac{m}{4\pi^2 r} \int_{-\infty}^{\infty} \left(\frac{1}{\varkappa - p} - \frac{1}{\varkappa + p} \right) \sin pr \, dp \overset{\text{pole integration.}}{=}$$

$$= -\frac{m}{2\pi r} e^{i \, \mathrm{Sign}\,\varepsilon \cdot (\varkappa r)},$$

where $\varkappa = \sqrt{2m(E_F + \varepsilon + i\delta \, \mathrm{Sign}\,\varepsilon)} \underset{\varepsilon \ll E_F}{\simeq} p_F + \frac{\varepsilon}{v_F}$

and $G(\varepsilon, r)\Big|_{\varepsilon \ll E_F} = -\frac{m}{2\pi r} \cdot e^{i \, \mathrm{sign}\,\varepsilon \cdot \left(p_F + \frac{\varepsilon}{v_F}\right)}$.

In 2D, the Green's function will read:

$$G(\varepsilon, \vec{r}) = \int_0^{\pi}\int_0^{\infty} \frac{e^{ipr\cos\theta}}{\varepsilon - \xi(\vec{p}) + i\delta \, \mathrm{Sign}\,\varepsilon} \, p \frac{dp}{2\pi} \frac{d\theta}{2\pi} =$$

$$= \frac{1}{2\pi} \int_0^{\infty} \frac{\pi J_0(pr)}{\varepsilon - \frac{p^2}{2m} + E_F + i\delta \, \mathrm{Sign}\,\varepsilon} \, p \frac{dp}{2\pi} \simeq$$

$$\underset{p_F r \gg 1}{\simeq} \frac{1}{\sqrt{2\pi p_F r}} e^{i \, \mathrm{Sign}\,\varepsilon \cdot \left(p_F + \frac{\varepsilon}{v_F}\right) + \frac{\pi i}{4}}$$

A heavy particle (impurity) in a Fermi-gas:

Consider a heavy particle of mass M in a Fermi-gas of particles with mass m: $M \gg m \Rightarrow$ the scattering of fermions is quasi-ellastic $\Rightarrow$ there is almost no energy exchange as the momentum acquired by the impurity is $\Delta p = 2k_F$ and the energy is

$$\varepsilon_M = \frac{\Delta p^2}{2M} = \frac{(2k_F)^2}{2M} \ll E_F = \frac{k_F^2}{2m}.$$

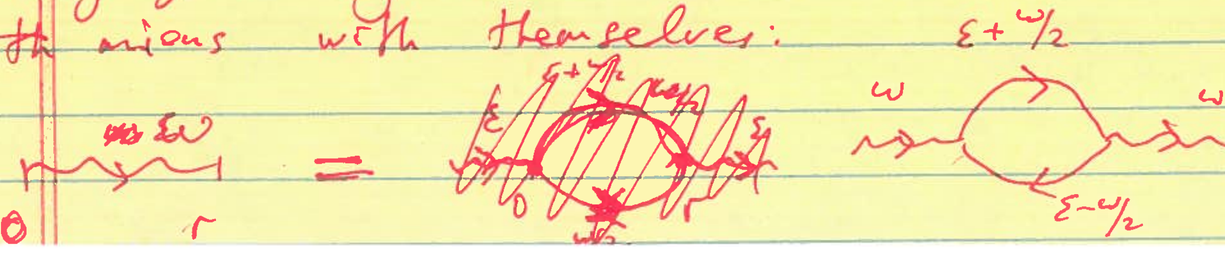
In the absence of scattering fermion Green's function reads:

$$G(\varepsilon, \vec{p}) = \frac{1}{\varepsilon - \vec{p}^{\,2}/2m + i\delta}.$$

Assume that the ~~electron~~ impurity gives rise to a weak contact interaction potential

$$V(\vec{r} - \vec{r}\,') = \lambda\, \delta(\vec{r} - \vec{r}\,').$$

Fermions are much faster $\Rightarrow$ fermion dynamics is effectively given by the retarded interaction of fer- ~~with the~~ mions with themselves:



This effective interaction is given by the polarization operator

$$\Pi(\omega, \vec{r}) = \underset{\substack{\downarrow \\ 2 \text{ spin species}}}{2}\, i \int G\left(\varepsilon + \frac{\omega}{2}, \vec{r}\right) \cdot G\left(\varepsilon - \frac{\omega}{2}, \vec{r}\right) \frac{d\varepsilon}{2\pi}.$$

In real-space coordinates, large $k \cdot r \gg 1$ behavior of the Green's function in 3D is given by

$$G(\varepsilon, r) = -\frac{m}{2\pi r} e^{i\, \text{sign}[\varepsilon] \cdot \varkappa(\varepsilon) \cdot r} \ * \ (\text{[illegible]}) \text{, derived this } \}$$

$$\varkappa(\varepsilon) = \sqrt{k_F^2 + 2m\varepsilon}.$$

We can also notice that the effective scattering potential $V_{eff}(\omega, \vec{r})$ must vanish at $\omega = 0$ (i.e. if there is no retardation in interactions).

$$\Rightarrow \quad V_{eff}(\omega, \vec{r}) = \lambda^2 \left[\Pi(\omega, \vec{r}) - \Pi(\omega = 0, \vec{r})\right],$$

$\boxed{\omega \ll E_F}$.

We have that $|\varepsilon| << E_F \Rightarrow \varkappa(\varepsilon)$ is almost constant $\Rightarrow$ $\varkappa = (k_F^2 + 2m\varepsilon)^{1/2} \simeq k_F$.

$$\Rightarrow V_{eff}(\omega, \vec{r}) = 2i\lambda^2 \int m^2 \sin^2(k_F r)\left[\operatorname{sign}\left(\varepsilon+\frac{\omega}{2}\right)\operatorname{sign}\left(\varepsilon-\frac{\omega}{2}\right) - 1\right] \cdot \frac{1}{(2\pi)^2 r^2} \frac{d\varepsilon}{2\pi} = -i|\omega| \cdot F(\vec{r})$$

where $F(\vec{r}) = \dfrac{\lambda^2 m^2}{2\pi^3 r^2} \sin^2(k_F r)$.

In k-space:

$$V_{eff}(\omega, \vec{k}) = -i|\omega| \int F(\vec{r})\, e^{-i\vec{k}\vec{r}}\, d^3\vec{r} = -i|\omega| \cdot \tilde{F}(\vec{k})$$

$$\tilde{F}(\vec{k}) = \frac{8\pi\lambda^2 m^2}{2\pi^3 |\vec{k}|} \int \sin^2(k_F r) \sin(|\vec{k}| r) \frac{dr}{r} = \frac{1}{\pi}\lambda^2 m^2 \cdot \begin{cases} \frac{1}{k}, & \text{if } |k| \le 2k_F \\ 0, & \text{if } |k| > 2k_F \end{cases} \Rightarrow$$

$\Rightarrow$ this means that at small energies $\omega << E_F$ there is no momentum transfer $> 2k_F$.

Here $r = |\vec{r}|$ -integration is performed using an identity

$$\int_0^\infty \sin^2(\alpha r)\,\sin(\beta r)\,\frac{dr}{r} = \frac{\pi}{8}\Big[2\,\mathrm{sign}\,\beta - \mathrm{sign}(\beta+2\alpha) - \mathrm{sign}(\beta-2\alpha)\Big].$$