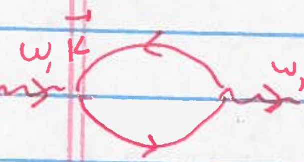
Density-density correlation function of the Fermi-gas:

The correlator is given by the following diagram:

$\Pi(\omega, k) =$ 

. Lets us calculate it at small

$$|\omega| \ll E_F, \quad |k| \ll k_F.$$

Writing analytical expression for Π we get

$$\Pi(\omega, k) = -2i \iint G_0\left(\varepsilon + \frac{\omega}{2}; \vec{q} + \frac{\vec{k}}{2}\right) G_0\left(\varepsilon - \frac{\omega}{2}; \vec{q} - \frac{\vec{k}}{2}\right) \frac{d^3 q}{(2\pi)^3} \frac{d\varepsilon}{2\pi}$$

-2 due to spine degrees.

$\vec{q} \pm \frac{\vec{k}}{2}$ - are in the vicinity of the F.S. since $\vec{q}$ is and

$\Rightarrow$ at $|\vec{k}| \ll k_F$:

$$\left|\vec{q} \pm \frac{\vec{k}}{2}\right| \simeq q \pm \frac{k}{2} \cos\theta, \quad \text{where} \quad \theta = \widehat{\vec{q}\,\vec{k}}.$$

Then
$$G_0\left(\varepsilon \pm \frac{\omega}{2}\right) = \frac{1}{\left(\varepsilon \pm \frac{\omega}{2}\right) - \xi\left(\left|\vec{q} \pm \frac{\vec{k}}{2}\right|\right) + i\delta \operatorname{sign}\left[\xi\left(\left|\vec{q} \pm \frac{\vec{k}}{2}\right|\right)\right]}$$

where $\xi(\vec{p}) = \frac{p^2}{2m} - E_F$ and thus

$$\xi\left(\left|\vec{q} \pm \frac{\vec{k}}{2}\right|\right) \simeq \xi\left(q \pm \frac{k}{2}\cos\theta\right) = \frac{\left(q \pm \frac{k}{2}\right)^2 \cos^2\theta}{2m} - E_F =$$

$$= \xi(q) \pm \frac{k v_F}{2} \cos\theta$$

Integration over ε in $\Pi(\omega,k)$ can be performed ~~done~~ by using pole integration and closing the contour in upper half-plane of ω and expanding G_0 into simple fractions. The integral is $\neq 0$ if both poles are in different half-plains:

$$\int \frac{d\varepsilon}{\left(\varepsilon+\frac{\omega}{2}-\xi_+ + i\delta\,\mathrm{sign}\,\xi_+\right)\left(\varepsilon+\frac{\omega}{2}-\xi_- + i\delta\,\mathrm{sign}\,\xi_-\right)} =$$

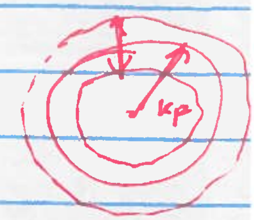
$$= 2\pi i \frac{n(\xi_-)-n(\xi_+)}{\omega - v_F k\cos\theta + i\delta(\mathrm{sign}\,\xi_+ - \mathrm{sign}\,\xi_-)},$$

where $\xi_\pm = \xi(q) \pm \frac{v_F k}{2}\cos\theta$ and $n(\xi) = \frac{1}{e^{(\xi-E_F)/k_B T}+1}$ is Fermi-Dirac distribution function. Since k is small $k \ll k_F \Rightarrow n(\xi_-)-n(\xi_+) \neq 0$ in a small vicinity of E_F $\Rightarrow$ one may perform integration over ξ. Depending on the sign of $\cos\theta$, one may have two situations:

1) $\cos\theta > 0 \Rightarrow n(\xi_-)-n(\xi_+) \neq 0$ when $|\xi| < \frac{v_F k}{2}\cos\theta$ and then $n(\xi_-)-n(\xi_+) = -1$

2) $\cos\theta < 0 \Rightarrow n(\xi_-)-n(\xi_+) \neq 0$ when $|\xi| < -\frac{v_F k}{2}\cos\theta$ and $n(\xi_-)-n(\xi_+) = 1$.

$$\frac{d^3 q}{(2\pi)^3} = q^2 \sin\theta \, \frac{dq}{2\pi} \, \frac{d\theta}{2\pi} \, \frac{d\varphi}{2\pi} \xrightarrow[\varphi \in (0,2\pi)]{} q^2 \sin\theta \, \frac{dq}{2\pi} \, \frac{d\theta}{2\pi}.$$



and integration over $q \in (0,\infty)$ yields

$$\Pi(\omega,k) = \nu_0 \int_0^{\pi} \frac{v_F k \cos k}{\omega - v_F k \cos\theta + i\delta \,\mathrm{Sign}\,\omega} \sin\theta \, d\theta, \quad \text{where}$$

$\nu_0 = m k_F \cdot \frac{1}{2\pi^2 \hbar^3}$ – is the density of states.

Let $x = \cos\theta$, then

$$\int_{-1}^{1} \frac{x\, dx}{x_0 - x + i\delta \,\mathrm{Sign}\, x_0} = A + iB, \quad \text{where}$$

$$A = -2 + x_0 \ln\left|\frac{x_0+1}{x_0-1}\right|, \quad B = \begin{cases} 0, & \text{if } |x_0| > 1 \\ -\pi x_0, & \text{if } 0 < x_0 < 1 \\ \pi x_0, & \text{if } -1 < x_0 < 0 \end{cases}.$$

Thus we obtain

$$\Pi(\omega,k) = -2\nu_0 \left[1 - \frac{\omega}{2kv_F} \ln\left|\frac{kv_F + \omega}{kv_F - \omega}\right| + \frac{\pi i |\omega|}{2kv_F} \, \theta\left[1 - \frac{|\omega|}{kv_F}\right] \right]$$

Now let us calculate the dynamical spin susceptibility: contribution to magnetic susceptibility $\chi(\omega, k)$ of electron gas at $T=0$, i.e. the responce of the system to an externally applied time-dependent magnetic field.

Take $\hat{A}(t) = \hat{B}(t) \longrightarrow \hat{S}_z(r,t)$, where in Kubo formula,

where $\hat{S}_z(r,t) = \mu_B \left[\hat{\psi}_\uparrow^+(r,t)\, \hat{\psi}_\uparrow(r,t) - \hat{\psi}_\downarrow^+(\vec{r},t)\, \hat{\psi}_\downarrow(\vec{r},t) \right]$:

$$\chi(\omega, k) = \frac{i}{\hbar} \int_0^\infty \langle [\hat{S}_z(r,t), \hat{S}_z(0,0)] \rangle e^{i\omega t}\, dt \Rightarrow$$

$$\chi(\omega,k) = 2 i \mu_B^2 \iint e^{i\omega t + i\vec{k}\vec{r}} \Big[\langle \hat{\psi}_\uparrow^+(r,t)\, \hat{\psi}_\uparrow(0,0)\rangle \langle \hat{\psi}_\uparrow(\vec{r},t)\, \hat{\psi}_\uparrow^+(0,0)\rangle -$$

due to spin

$$- \langle \hat{\psi}_\uparrow^+(0,0)\, \hat{\psi}_\uparrow(\vec{r},t)\rangle \langle \hat{\Psi}_\uparrow(0,0)\, \hat{\Psi}_\uparrow^+(\vec{r},t)\rangle \Big] dt\, d^3r,$$

where we used the Wick's theorem.

The Green's functions are given by:

$$\langle \hat{\psi}_\uparrow^+(\vec{r},t)\, \hat{\psi}_\uparrow(\vec{r}\,',t')\rangle = \sum_{\vec{p}} e^{-i\xi(\vec{p})\cdot(t'-t) + i\vec{p}(\vec{r}\,'-\vec{r}\,')} \cdot n(\xi[\vec{p}])$$

$$\langle \hat{\psi}_\uparrow(\vec{r},t)\, \hat{\psi}_\uparrow^+(\vec{r}\,',t')\rangle = \sum_{\vec{p}} e^{-i\xi(\vec{p})(t'-t) + i\vec{p}(\vec{r}\,'-\vec{r})} [1 - n[\xi(\vec{p})]]$$

where $n(\xi)$ is the Fermi-Dirac function.

Performing Fourier transformation, one obtains

$$\chi(\omega, E) = 2i\mu_B^2 \int_{|q_+|>k_F} \int_{|q_-|<k_F} \left[G^R(\varepsilon_+, \vec{q}_+)\, G^A(\varepsilon_-, \vec{q}_-) - \right.$$

$$\left. - G^A(\varepsilon_-, \vec{q}_+)\, G^R(\varepsilon_+, \vec{q}_-) \right] \frac{d\varepsilon\, d^3q}{(2\pi)^4}, \text{ where}$$

$$G^{R/A}(\varepsilon, \vec{q}) = \frac{1}{\varepsilon - \xi(\vec{q}) \pm i\delta}$$

Energy integration yields: $\int G^R(\varepsilon_+, \vec{q}_+)\, G^A(\varepsilon_-, \vec{q}_-)\, d\varepsilon =$

$$\cong \frac{2\pi i}{\omega - v_F k \cos\theta + i\delta}, \text{ where } \theta = \vec{k}\,\widehat{\ }\,\vec{q}.$$

Therefore, one obtains

$$\chi_{\omega,k} = -\frac{\mu_B^2}{4\pi^2} \int_{\substack{|q_+|>k_F \\ |q_-|<k_F}} \left[\frac{1}{\omega - v_F k \cos\theta + i\delta} - \frac{1}{\omega + v_F k \cos\theta + i\delta} \right] d^3q$$

- the meaning is clear: fluctuating external field excites an electron-hole pair. Electron's momentum is q_+, and the hole has momentum q_-.

Because of Pauli principle one has a phase space restriction: $|q_+| > k_F$, $|q_-| < k_F \Rightarrow \cos\theta > 0$.

The energy of the pair is $\xi(q_+) - \xi(q_-) = v_F k \cos\theta$ and since $k \ll k_F$, these energies are small.

$kv_F \ll E_F$ - is the scale for particle-hole energy.

One can integrate over momenta, which just gives the phase space. The next step is to integrate over $\theta \in (0, \pi/2)$ [as the energy of particle-hole excitation > 0].

Notice that the change of $\omega + i\delta \to -\omega - i\delta$ is equivalent to $\theta \to \frac{\pi}{2} - \theta$.

$$\Rightarrow \quad \chi_{\omega,k} = \mu_B^2 \nu_0 \int_0^{\pi} \frac{v_F k \cos\theta}{\omega - v_F k \cos\theta + i\delta} \sin\theta \, d\theta$$

Introducing integration variable: $x = \cos\theta$:

$$\int_{-1}^{1} \frac{x\,dx}{x_0 - x + i\delta} = -2 + x_0 \ln\left(\frac{x_0 + i\delta + 1}{x_0 + i\delta - 1}\right),$$

where $x_0 = \frac{\omega}{kv_F}$. From here it follows

$$\chi_{\omega,k}^{*} = 2\mu_B^2 \nu_0 \left[1 - \frac{\omega}{2kv_F} \ln\left| \frac{kv_F+\omega}{kv_F-\omega} \right| + \frac{\pi i}{2} \frac{\omega}{kv_F} \theta\left(1 - \frac{|\omega|}{kv_F}\right) \right].$$

In statistical limit: $\frac{\omega}{kv_F} \to 0$, $k \to 0$ $\Rightarrow$ one obtains Pauli's susceptibility

$$\chi_{paramagnetic} = 2\mu_B^2 \nu_0, \qquad \nu_0 = \frac{mk_F}{2\pi^2\hbar^3}$$

$\chi_p = \frac{M}{H}$, where M - magnetisation $= n_\downarrow - n_\uparrow$,
H - z-directional magnetic field.

Density-density responce in 1D

Consider a 1D fermi gas in an external fluctuating field,

$$\hat{H}_{int}(t) = -\int \varphi(x,t)\, \hat{n}(x,t)\, dx$$ and

consider the linear responce of $\hat{n}(x,t)$-density on $\varphi(x,t)$-field:

$$\langle n(x,t) \rangle = \iint_{t'<t} Q(x-x', t-t')\, \varphi(x',t')\, dx'dt',$$ or

in Fourier representation

$$\langle \hat{n}_{k,\omega} \rangle = Q(k,\omega) \cdot \varphi_{k,\omega}$$

The responce function $Q(k,\omega)$ can be evaluated using Kubo formula.

For spinless electrons

$$\hat{n}(x,t) = \hat{\psi}^{+}(x,t)\,\hat{\psi}(x,t)$$ and thus

$$Q(\omega,k) = \frac{i}{\hbar}\int_0^\infty \langle [\hat{n}_k(t), \hat{n}_k(0)] \rangle e^{i\omega t}\, dt$$

Using Wick's decoupling as in previous case, we obtain

$$Q(\omega,k) = \int_{\substack{|q+\frac{k}{2}|>k_F \\ |q-\frac{k}{2}|<k_F}} \left(\frac{1}{\omega - \frac{qk}{m} + i\delta} + \frac{1}{-\omega - qk/m - i\delta} \right) \frac{dq}{2\pi}$$

Due to Pauli principel one has phase-space restrictions: At small $k \ll k_F$, one can take

$q = k_F$ at $k>0$ and $q = -k_F$ at $k<0$. This is equivalent to linearization of dispersion

$$\xi(p) = \frac{p^2}{2m} - E_F \rightarrow \frac{(|p|-k_F)(|p|+k_F)}{2m} \simeq v_F(|p|-k_F).$$

$$q_\pm = q \pm K/2.$$

The vertex is

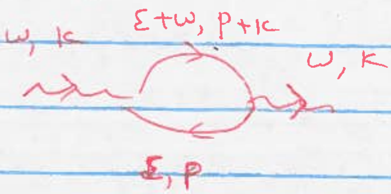
$$Q(\omega, k) = \frac{k}{2\pi}\left(\frac{1}{\omega - v_F|k| + i\delta} - \frac{1}{\omega + v_F|k| + i\delta}\right), \text{ or}$$

phase-space restriction

$$Q(\omega, k) = \frac{V_{1D} v_F^2 k^2}{\omega^2 - v_F^2 k^2 + i\delta \,\mathrm{sign}\, \omega}, \quad \text{where } V_{1D} = \frac{1}{\pi v_F}.$$

Polarization operator in 1D

The analytical expression corresponding to the diagram

$\Pi(\omega, k) =$ 

is given by:

$$\Pi(\omega,k) = -2ig^2 \int G(p,\varepsilon)\, G(p+k, \varepsilon+\omega)\, \frac{dp\, d\varepsilon}{(2\pi)^2}$$

↳ 2 corresponds to electron spin.

Integration over ε (as before) yields:

$$\Pi(\omega,k) = -\frac{g^2}{\pi} \int_{-\infty}^{\infty} \frac{[n(\xi_p) - n(\xi_{p+k})]\, dp}{\omega - \xi_{p+k} + \xi_p + i\delta\, \lambda_{p,k}},$$

where $\lambda_{p,k} = \text{sign}\, \xi_{p+k} - \text{sign}\, \xi_p$, and $n(\xi_p)$ is Fermi-distribution function.

The difference $[n(\xi_p) - n(\xi_{p+k})] \neq 0$ when

a) $\xi_p > 0$, $\xi_{p+k} < 0$;

b) $\xi_p < 0$, $\xi_{p+k} > 0$.

Consider $k > 0 \Rightarrow$ ~~reg~~ combined region of (a) and (b) is

a) $-k_F - k < p < -k_F$

b) $k_F - k < p < k_F$.

Given this, the expression for $\Pi(\omega, k)$ can be re-cast as follow:

$$\Pi(\omega,k) = -\frac{g^2}{\pi}\int_{-k_F}^{k_F} \frac{dp}{\omega - k^2/2m - \frac{pk}{m} - i\delta} +$$

$$+ \frac{g^2}{\pi}\int_{k_F-k}^{k_F} \frac{dp}{\omega - k^2/2m - pk/m + i\delta}.$$

Integration over p is straightforward; It gives:

$$\Pi(\omega,k) = \frac{mg^2}{\pi k}\ln X,$$

where

$$X = \frac{\left(\frac{k^2}{2m} - \frac{kk_F}{m} - \omega + i\delta\right)\left(\frac{k^2}{2m} - \frac{kk_F}{m} + \omega + i\delta\right)}{\left(\frac{k^2}{2m} + \frac{kk_F}{m} + \omega - i\delta\right)\left(\frac{k^2}{2m} + \frac{kk_F}{m} - \omega - i\delta\right)}$$

Now consider $\Pi(\omega,k)$ near $2k_F$: take $k = 2k_F + x$, and $\omega = 0$ $\Rightarrow$

$$\Rightarrow \boxed{\Pi(\omega=0,\ k=2k_F+x) = -\frac{mg^2}{\pi k_F}\cdot\ln\frac{k_F}{|x|}}.$$

We see that the polarization operator diverges near $k = 2k_F$ and at small ω.

~~In this region, the~~ This logarithmic singularity shows up in phonon spectrum. Phonon dispersion relation can be found from dressed (renormalized) propagator:

$$D^{-1}(\omega, k) = D_0^{-1}(\omega, k) - \Pi(\omega, k)$$

↳ bare phonon propagator

upon equating $D^{-1}(\omega, k) = 0$. From here we see that,

$$\frac{\omega^2 - \omega_{2k_F}^2}{\omega_{2k_F}^2} + \frac{mg^2}{\pi k_F} \ln \frac{k_F}{|k - 2k_F|} = 0, \quad \omega_{2k_F}^2 = (c \cdot 2k_F$$

Strictly speaking, one should have used here Π_{2k_F} at finite ω, but this does not change the result:

Dispersion of phonons in 1D becomes

$$\boxed{\omega^2 = \omega_{2k_F}^2 \left(1 - \frac{mg^2}{\pi k_F} \ln \frac{k_F}{|k - 2k_F|}\right)}$$

$\Rightarrow$ RHS becomes negative at $k \sim 2k_F$ and ω - becomes imaginary $\Rightarrow$

$\Rightarrow$ INSTABILITY.