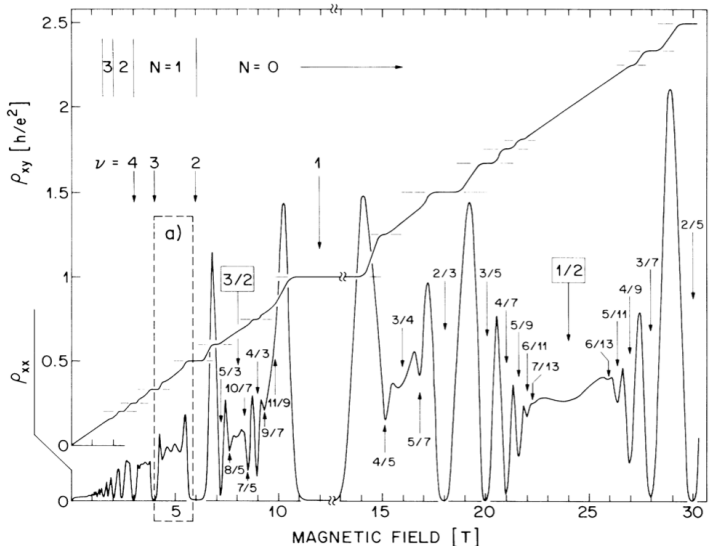
3. The Fractional Quantum Hall Effect

We've come to a pretty good understanding of the integer quantum Hall effect and the reasons behind it's robustness. Indeed, some of the topological arguments in the previous chapter are so compelling that you might think the Hall resistivity of an insulator has to be an integer. But each of these arguments has a subtle loophole and ultimately they hold only for non-interacting electrons. As we will now see, much more interesting things can happen when we include interactions.

As with the integer quantum Hall effect, these interesting things were first discovered by experimenters rather than theorists. Indeed, it came as a great surprise to the community when, in 1982, plateaux in the Hall resistivity were seen at non-integer filling fractions. These plateaux were first seen at filling fraction $\nu = \frac{1}{3}$ and $\frac{2}{3}$, and later at $\nu = \frac{1}{5}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{9}, \frac{3}{5}, \ldots$ in the lowest Landau level and $\nu = \frac{4}{3}, \frac{5}{3}, \frac{7}{5}, \frac{5}{2}, \frac{12}{5}, \frac{13}{5}, \ldots$ in higher Landau levels, as well as many others. There are now around 80 quantum Hall plateaux that have been observed. A number of these are shown below[13]:



There's two things that we can say immediately. First, the interactions between electrons must be playing some role. And second, the answer to why these plateaux form is likely to be very hard. Let's see why.

Suppose, for the sake of argument, that we have $\nu < 1$ so that the lowest Landau level is partially filled. Each Landau level can house $\mathcal{N} = AB/\Phi_0$ (spin polarised)

[13]This data is from R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard and H. English "*Observation of an Even-Denominator Quantum Number in the Fractional Quantum Hall Effect*", Phys. Rev. Lett. 59, 15 (1987).

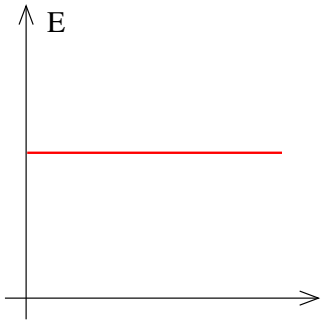


Figure 25: Density of states in the lowest Landau level without interactions

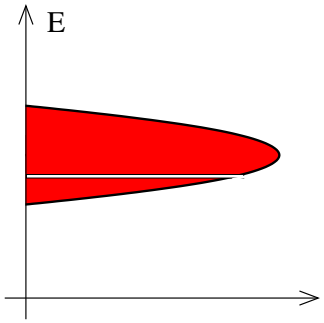


Figure 26: ...and with interactions (with only a single gap at $\nu = 1/3$ shown.

electrons, where B is the magnetic field and A is the area of the sample. This is a macroscopic number of electrons. The number of ways to fill $\nu\mathcal{N}$ of these states is $\binom{\mathcal{N}}{\nu\mathcal{N}}$ which, using Stirling's formula, is approximately $\left(\frac{1}{\nu}\right)^{\nu\mathcal{N}}\left(\frac{1}{1-\nu}\right)^{(1-\nu)\mathcal{N}}$. This is a ridiculously large number: an exponential of an exponential. The ground state of any partially filled Landau level is wildly, macroscopically degenerate.

Now consider the effect of the Coulomb interaction between electrons,

$$V_{\rm Coulomb} = \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_i - \mathbf{r}_j|} \tag{3.1}$$

On general grounds, we would expect that such an interaction would lift the degeneracy of ground states. But how to pick the right one? The approach we're taught as undergraduates is to use perturbation theory. But, in this case, we're stuck with extraordinarily degenerate perturbation theory where we need to diagonalise a macroscopically large matrix. That's very very hard. Even numerically, no one can do this for more than a dozen or so particles.

We can, however, use the experiments to intuit what must be going on. As we mentioned above, we expect the the electron interactions to lift the degeneracy of the Landau level, resulting in a spectrum of states of width $\sim E_{\rm Coulomb}$. The data would be nicely explained if this spectrum had gaps at the filling fractions ν where Hall states are seen. In the picture above, we've depicted just a single gap at $\nu = 1/3$. Presumably though there are many gaps at different fractions: the more prominent the plateaux, the larger the gap.

Then we can just re-run the story we saw before: we include some disorder, which introduces localised states within the gap, which then gives rise both to the plateaux

in ρ_{xy} and the observed $\rho_{xx} = 0$. The bigger the gap, the more prominent the observed plateaux. This whole story requires the hierarchy of energy scales,

$$\hbar\omega_B \gg E_{\rm Coulomb} \gg V_{\rm disorder}$$

We will assume in what follows that this is the case. The question that we will focus on instead is: what is the physics of these fractional quantum Hall states?

In what follows, we will take advantage of the difficulty of a direct theoretical attack on the problem to give us license to be more creative. As we'll see, the level of rigour in the thinking about the fractional quantum Hall effect is somewhat lower than that of the integer effect. Instead, we will paint a compelling picture, using a number of different approaches, to describe what's going on.

3.1 Laughlin States

The first approach to the fractional quantum Hall effect was due to Laughlin[14], who described the physics at filling fractions

$$\nu = \frac{1}{m}$$

with m an odd integer. As we've explained above, it's too difficult to diagonalise the Hamiltonian exactly. Instead Laughlin did something very bold: he simply wrote down the answer. This was motivated by a combination of physical insight and guesswork. As we will see, his guess isn't exactly right but, it's very close to being right. More importantly, it captures all the relevant physics.

3.1.1 The Laughlin Wavefunction

Laughlin's wavefunction didn't come out of nowhere. To motivate it, let's start by considering an illuminating toy model.

Two Particles

Consider two particles interacting in the lowest Landau level. We take an arbitrary central potential between them,

$$V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$$

Recall that in our first courses on classical mechanics we solve problems like this by using the conservation of angular momentum. In quantum physics, this means that we

[14]The original paper is "*Anomalous quantum Hall effect: An Incompressible quantum fluid with fractionally charged excitations*" , Phys. Rev. Lett. 50 (1983) 1395.

work with eigenstates of angular momentum. As we saw in Section 1.4, if we want to talk about angular momentum in Landau levels, we should work in symmetric gauge. The single particle wavefunctions in the lowest Landau level take the form (1.30)

$$\psi_m \sim z^m e^{-|z|^2/4l_B^2}$$

with $z = x - iy$. These states are localised on a ring of radius $r = \sqrt{2m}l_B$. The exponent m of these wavefunctions labels the angular momentum. This can be seen by acting with the angular momentum operator (1.31),

$$J = i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \hbar(z\partial - \bar{z}\bar{\partial}) \qquad \Rightarrow \qquad J\psi_m = \hbar m\,\psi_m$$

Rather remarkably, this information is enough to solve our two-particle problem for any potential V! As long as we neglect mixing between Landau levels (which is valid if $\hbar\omega_B \gg V$) then the two-particle eigenstates for any potential must take the form

$$\psi = (z_1 + z_2)^M (z_1 - z_2)^m e^{-(|z_1|^2+|z_2|^2)/4l_B^2}$$

where M, m are non-negative integers, with M determining the angular momentum of the centre of mass, and m the relative angular momentum. Note that here, and below, we've made no attempt to normalise the wavefunctions.

It's surprising that we can just write down eigenfunctions for a complicated potentials $V(r)$ without having to solve the Schrödinger equation. It's even more surprising that all potentials $V(r)$ have the same energy eigenstates. It is our insistence that we lie in the lowest Landau level that allows us to do this.

Many-Particles

Unfortunately, it's not possible to generalise arguments similar to those above to uniquely determine the eigenstates for $N > 2$ particles. Nonetheless, on general grounds, any lowest Landau level wavefunction must take the form,

$$\psi(z_1, \ldots, z_n) = f(z_1, \ldots, z_N) e^{-\sum_{i=1}^N |z_i|^2/4l_B^2} \tag{3.2}$$

for some analytic function $f(z)$. Moreover, this function must be anti-symmetric under exchange of any two particle $z_i \leftrightarrow z_j$, reflecting the fact that the underlying electrons are fermions.

Laughlin's proposal for the ground state wavefunction at filling fraction $\nu = 1/m$ is:

$$\psi(z_i) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_{i=1}^n |z_i|^2/4l_B^2} \tag{3.3}$$

Clearly this is anti-symmetric when m is an odd integer. For m an even integer, it can be thought of as a quantum Hall state for bosons. The pre-factor vanishes with a zero of order m whenever two electrons come together. Meanwhile, the exponential factor decreases quickly whenever the electrons get too far away from the origin. The wavefunction is peaked on configurations that balance these two effects.

Let's first show that the wavefunction has the desired filling fraction. To do this, focus on what the wavefunction is telling us about a single particle, say z_1. The terms that depend on z_1 in the pre-factor of the Laughlin wavefunction are

$$\prod_{i<j}(z_i - z_j)^m \sim \prod_{i=2}^{N}(z_1 - z_i)^m$$

which tells us that there are $m(N-1)$ powers of z_1. This, in turn, tells us that the maximum angular momentum of the first particle is $m(N-1)$ and so its maximum radius is $R \approx \sqrt{2mN}l_B$. Correspondingly, the area of the droplet is $A \approx 2\pi mNl_B^2$ (where we've replaced $N-1$ with N). Recall that the number of states in the full Landau level is $AB/\Phi_0 = A/2\pi l_B^2 \approx mN$. This argument gives us the filling fraction

$$\nu = \frac{1}{m} \tag{3.4}$$

as promised.

It can be shown numerically that, at least for small numbers of particles, this wavefunction has greater than 99% overlap with the true ground state arising arising from both the Coulomb repulsion (3.1) as well as a number of other repulsive potentials V. Heuristically this occurs because the wavefunction has a zero of order m whenever two electrons coincide. Of course, a single zero is guaranteed by Pauli exclusion, but the Laughlin wavefunction does more. It's as if each electron carves out a space around it which helps it minimise the for repulsive potentials.

The high numerical overlap with the true ground state is often put forward as strong evidence for the veracity of the Laughlin wavefunction. While it's certainly impressive, this isn't the reason that the Laughlin wavefunction is interesting. Finding the ground state numerically is difficult and can only be done for a couple of dozen particles. While this may provide 99.99% overlap with the Laughlin wavefunction, by the time we get

to 10^{11} particles or so, the overlap is likely to be essentially zero. Instead, we should think of the Laughlin wavefunctions as states which lie in the same "universality class" as the true ground state. As we explain in more detail in Section 3.2, this means that the states have the same fractional excitations and the same topological order.

The Fully Filled Landau Level

From the arguments above, the Laughlin state (3.3) with $m = 1$ should describe a completely filled Landau level. But this is something we can compute in the non-interacting picture and it provides a simple check on the Laughlin ansatz.

Let us first review how to build the many-particle wavefunction for non-interacting electrons. Suppose that N electrons sit in states $\psi_i(x)$, with $1 = 1, \ldots, N$. Because the electrons are fermions, these states must be distinct. To build the many-particle wavefunction, we need to anti-symmetrise over all particles. This is achieved by the *Slater determinant*,

$$\psi(x_i) = \begin{vmatrix} \psi_1(x_1) & \psi_1(x_2) & \ldots & \psi_1(x_N) \\ \psi_2(x_1) & \psi_2(x_2) & \ldots & \psi_2(x_N) \\ \vdots & & & \vdots \\ \psi_N(x_1) & \psi_N(x_2) & \ldots & \psi_N(x_N) \end{vmatrix} \tag{3.5}$$

We can now apply this to the lowest Landau level, with the single-particle states built up with successive angular momentum quantum numbers

$$\psi_m(z) \sim z^{m-1} e^{-|z|^2/4l_B^2} \qquad m = 1, \ldots, N$$

The resulting Slater determinant gives a state of the general form (3.2), with $f(z_i)$ given by a function known as the *Vandermonde determinant*,

$$f(z_i) = \begin{vmatrix} z_1^0 & z_2^0 & \ldots & z_N^0 \\ z_1 & z_2 & \ldots & z_3 \\ \vdots & & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_N^{N-1} \end{vmatrix} = \prod_{i<j} (z_i - z_j)$$

To see that the determinant is indeed given by the product factor, note that $\prod_{i<j}(z_i - z_j)$ is the lowest order, fully anti-symmetric polynomial (because any such polynomial must have a factor $(z_i - z_j)$ for each pair $i \neq j$). Meanwhile, the determinant is also completely anti-symmetric and has the same order as the product factor. This ensures that they must be equal up to an overall numerical factor which can be checked to be 1. We see that $m = 1$ Laughlin state does indeed agree with the wavefunction for a completely filled lowest Landau level.

The Competing Phase: The Wigner Crystal

The Laughlin state should be thought of as a liquid phase of electrons. In fact, strictly speaking, it should be thought of as an entirely new phase of matter, distinguished by a property called *topological order* which we'll discuss in Section 3.2.4. But, if you're looking for a classical analogy, a liquid is the best.

There is a competing solid phase in which the electrons form a two-dimensional triangular lattice, known as a *Wigner crystal*. Indeed, before the discovery of the quantum Hall effect, it was thought that this would be the preferred phase of electrons in high magnetic fields. It's now known that the Wigner crystal has lower energy than the Laughlin state only when the densities of electrons are low. It is observed for filling fractions $\nu \lesssim \frac{1}{7}$

3.1.2 Plasma Analogy

The Laughlin wavefunctions (3.3) are very easy to write down. But it's hard to actually compute with them. The reason is simple: they are wavefunctions for a macroscopic number of particles which means that if we want to compute expectation values of operators, we're going to have to do a macroscopic number of integrals $\int d^2z_i$. And that's difficult.

For example, suppose that we want to figure out the average density of the quantum Hall droplet. We need to compute the expectation value of the density operator

$$n(z) = \sum_{i=1}^{N} \delta(z - z_i)$$

This is given by

$$\langle \psi | n(z) | \psi \rangle = \frac{\int \prod_{i=1}^{N} d^2z_i \; n(z) P[z_i]}{\prod_{i=1}^{N} d^2z_i \; P[z_i]} \tag{3.6}$$

where we've introduced the un-normalised probability density associated to the Laughlin wavefunction

$$P[z_i] = \prod_{i<j} \frac{|z_i - z_j|^{2m}}{l_B^{2m}} e^{-\sum_i |z_i|^2/2l_B^2} \tag{3.7}$$

The integrals in (3.6) are hard. How to proceed?

The key observation is that the expectation value (3.6) has the same formal structure as the kind of things we compute in classical statistical mechanics, with the denominator interpreted as the partition function,

$$Z = \prod_{i=1}^{N} d^2 z_i \; P[z_i]$$

Indeed, we can make this analogy sharper by writing the probability distribution (3.7) so it looks like a Boltzmann distribution function,

$$P[z_i] = e^{-\beta U(z_i)}$$

with

$$\beta U(z_i) = -2m \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{1}{2l_B^2}\sum_{i=1}^{N} |z_i|^2$$

Of course, this hasn't helped us do the integrals. But the hope is that perhaps we can interpret the potential $U(z_i)$ as something familiar from statistical physics which will at least provide us with some intuition for what to expect. And, indeed, this does turn out to be the case, but only if we pick β — which, in a statistical mechanics context is interpreted as inverse temperature — to take the specific value

$$\beta = \frac{2}{m} \tag{3.8}$$

I stress that the quantum Hall state isn't placed at a finite temperature. This is an auxiliary, or fake, "temperature". Indeed, you can tell it's not a real temperature because it's dimensionless! To compensate, the potential is also dimensionless, given by

$$U(z_i) = -m^2 \sum_{i<j} \log\left(\frac{|z_i - z_j|}{l_B}\right) + \frac{m}{4l_B^2}\sum_{i=1}^{N} |z_i|^2 \tag{3.9}$$

We'll now show that this is the potential energy for a plasma of charged particles moving in two-dimensions, where each particle carried electric charge $q = -m$.

The first term in (3.9) is the Coulomb potential between two particles of charge q when both the particle and the electric field lines are restricted to lie in a two-dimensional plane. To see this, note that Poisson equation in two dimensions tells us that the electrostatic potential generated by a point charge q is

$$-\nabla^2\phi = 2\pi q\delta^2(\mathbf{r}) \quad \Rightarrow \quad \phi = -q\log\left(\frac{r}{l_B}\right)$$

The potential energy between two particles of charge q is then $U = q\phi$, which is indeed the first term in (3.9).

The second term in (3.9) describes a neutralising background of constant charge. A constant background of charge density ρ_0 would have electrostatic potential obeying $-\nabla^2\phi = 2\pi\rho_0$. Meanwhile, the second term in the potential obeys

$$-\nabla^2\left(\frac{|z|^2}{4l_B^2}\right) = -\frac{1}{l_B^2}$$

which tells us that each electron feels a background charge density

$$\rho_0 = -\frac{1}{2\pi l_B^2} \tag{3.10}$$

Note that this is equal (up to fundamental constants) to the background flux B in the quantum Hall sample.

Now we can use our intuition about this plasma. To minimise the energy, the plasma will want to neutralise, on average, the background charge density. Each particle carries charge $q = -m$ which means that the compensating density of particles n should be $mn = \rho_0$, or

$$n = \frac{1}{2\pi l_B^2 m}$$

This is the expected density of a state at filling fraction $\nu = 1/m$. This argument has also told us something new. Naively, the form of the Laughlin wavefunction makes it look as if the origin is special. But that's misleading. The plasma analogy tells us that the average density of particles is constant.

The plasma analogy can also help answer more detailed questions about the variation of the density (3.6) on shorter distance scales. Intuitively, we might expect that at low temperatures (keeping the density fixed), the plasma forms a solid, crystal like structure, while at high temperatures it is a liquid. Alternatively, at low densities (keeping the temperature fixed) we would expect it to form a solid while, at high densities, it would be a liquid. To determine the structure of the Laughlin wavefunction, we should ask which phase the plasma lies in at temperature $\beta = 2/m$ and density $n = 1/2\pi l_B^2 m$.

This is a question which can only be answered by numerical work. It turns out that the plasma is a solid when $m \gtrsim 70$. For the low m of interest, in particular $m = 3$ and 5, the Laughlin wavefunction describes a liquid. (Note that this is not the same issue as whether the Wigner crystal wavefunction is preferred over the Laughlin wavefunction: it's a question of whether the Laughlin wavefunction itself describes a liquid or solid).

3.1.3 Toy Hamiltonians

The Laughlin state (3.3) is not the exact ground state of the Hamiltonian with Coulomb repulsion. However, it is possible to write down a toy Hamiltonian whose ground state is given by the Laughlin state. Here we explain how to do this, using some tools which will also provide us with a better understanding of the general problem.

Let's go back to the problem of two particles interacting through a general central potential $V(|\mathbf{r}_1 - \mathbf{r}_2|)$. As we saw in Section 3.1.1, in the lowest Landau level the eigenstates for any potential are the same, characterised by two non-negative integers: the angular momentum of the centre of mass M and the relative angular momentum m,

$$|M, m\rangle \sim (z_1 + z_2)^M (z_1 - z_2)^m e^{-(|z_1|^2+|z_2|^2)/4l_B^2}$$

We should take m odd if the particles are fermions, m even if they are bosons.

The eigenvalues of the potential V are given by

$$v_m = \frac{\langle M, m|V|M, m\rangle}{\langle M, m|M, m\rangle} \tag{3.11}$$

These eigenvalues are sometimes referred to as *Haldane pseudopotentials*. For central potentials, they do not depend on the overall angular momentum M.

These eigenvalues capture a crude picture of the spatial profile of the potential. This is because, as we have seen, the wavefunctions $|M, m\rangle$ are peaked on a circle of radius $r \approx \sqrt{2m}l_B$. Correspondingly, the eigenvalues are roughly

$$v_m \approx V(r = \sqrt{2m}l_B) \tag{3.12}$$

This means that typically the v_m are positive for a repulsive potential and negative for an attractive potential, in each case falling off as $V(r)$ as m increases.

Importantly, however, the eigenvalues are discrete. This simple fact is telling us some interesting physics: it means that each of the states $|M, m\rangle$ can be thought of as a bound state of two particles, even if the potential is repulsive! This is in stark contrast to quantum mechanics in the absence of a magnetic field where there are no discrete-energy bound states for a repulsive potential, only scattering states with a continuous spectrum. But the magnetic field changes this behaviour.

Given the eigenvalues v_m, we can always reconstruct the potential V. In this lowest Landau level, there is no kinetic energy and the potential is the only contribution to the Hamiltonian. It's useful to write it as

$$H = \sum_{m'} v_{m'} \mathcal{P}_{m'} \tag{3.13}$$

where $\mathcal{P}_m$ is the operator which projects onto states in which the two particles have relative angular momentum m.

Now we can just pick whatever v_m we like to design our own Hamiltonians. Of course, they may not be very realistic when written in terms of $V(r)$ but we won't let that bother us too much. In this spirit, consider the choice

$$v_{m'} = \begin{cases} 1 & m' < m \\ 0 & m' \geq m \end{cases} \tag{3.14}$$

This Hamiltonian means that you pay an energy cost if the relative angular momentum of the particles dips below some fixed value m. But it costs you nothing to have a high angular momentum. In position space, the equation (3.12) tells us that there's a finite energy cost if the particles get too close.

Toy Hamiltonians for Many Particles

We can also use the pseudopotentials to construct Hamiltonians for N particles. To do this, we introduce the operator $\mathcal{P}_m(ij)$. This projects the wavefunction onto the state in which the i^{th} and j^{th} particles have relative angular momentum m. We then construct the Hamiltonian as

$$H = \sum_{m'=1}^{\infty} \sum_{i<j} v_{m'} \mathcal{P}_{m'}(ij) \tag{3.15}$$

Note, however, that $\mathcal{P}_m(ij)$ and $\mathcal{P}_m(kj)$ do not commute with each other. This is what makes these many-particle Hamiltonians difficult to solve.

Now consider the many-particle Hamiltonian with $v_{m'}$ given by (3.14). This time, you pay an energy cost whenever the relative angular momentum of any pair of particles is less than m. You can avoid this energy cost by including a factor of $(z_i - z_j)^m$ for each pair of particles, and writing down a wavefunction of the form

$$\psi(z_i) = s(z_i) \prod_{i<j} (z_i - z_j)^m \, e^{-\sum_i |z_i|^2/4l_B^2} \tag{3.16}$$

where $s(z_i)$ can be any symmetric polynomial in the z_i to preserve the statistics of the particles. All such wavefunctions have the vanishing energy.

So far this doesn't pick out the Laughlin state, which has $s(z_i) = 1$, as the ground state. But there is something special about this state: among all states (3.16), it is the most compact. Indeed, we saw in Section 3.1.1 that it takes up an area $A = 2\pi mNl_B^2$. Any state with $s(z_i) \neq 1$ necessarily spreads over a larger spatial area. This means that the Laughlin wavefunction will be the ground state if we also add a confining potential to the system.

We can state this in a slightly different way in terms of angular momentum. We know that states with higher angular momentum sit at larger radius. This means that we can take the total angular momentum operator J as a proxy for the confining potential and consider the Hamiltonian

$$H = \sum_{m'=1}^{m-1} \sum_{i<j} \mathcal{P}_{m'}(ij) + \omega J \tag{3.17}$$

The Laughlin wavefunction has the lowest energy: $E_0 = \frac{1}{2}\omega mN(N-1)$. Any wavefunction of the form (3.16) with $s(z_i) \neq 1$ has spatial extent larger than the ground state, and hence higher angular momentum, and so costs extra energy due to the second term; any wavefunction with spatial extent smaller than the Laughlin wavefunction necessarily has a pair of particles with relative angular momentum less than m and so pays an energy cost due to the first term.

The fact that it costs a finite energy to squeeze the wavefunction is expected to hold for more realistic Hamiltonians as well. It is usually expressed by saying that the quantum Hall fluid is *incompressible*. This is responsible for the gap in the bulk spectrum described in the introduction of this section. However, it turns out that the dynamics of states with $s(z_i) \neq 1$ contains some interesting information. We'll return to this in Section 6.1.

3.2 Quasi-Holes and Quasi-Particles

So far, we've only discussed the ground state of the $\nu = 1/m$ quantum Hall systems. Now we turn to their excitations. There are two types of charged excitations, known as *quasi-holes* and *quasi-particles*. We discuss them in turn.

Quasi-Holes

The wavefunction describing a quasi-hole at position $\eta \in \mathbf{C}$ is

$$\psi_{\rm hole}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta) \prod_{k<l}(z_k - z_l)^m \, e^{-\sum_{i=1}^n |z_i|^2/4l_B^2} \tag{3.18}$$

We see that the electron density now vanishes at the point η. In other words, we have created a "hole" in the electron fluid. More generally, we can introduce M quasi-holes in the quantum Hall fluid at positions η_j with $j = 1, \ldots, M$, with wavefunction

$$\psi_{\rm M-hole}(z;\eta) = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m\, e^{-\sum_{i=1}^n |z_i|^2/4l_B^2} \tag{3.19}$$

The quasi-hole has a remarkable property: it carries a fraction of the electric charge of the electron! In our convention, the electron has charge $-e$; the quasi-hole has charge $e^* = +e/m$.

A heuristic explanation of the fractional charge follows from noting that if we place m quasi-holes at the same point η then the wavefunction becomes

$$\psi_{\rm m-hole}(z;\eta) = \prod_{i=1}^{N}(z_i - \eta)^m\prod_{k<l}(z_k - z_l)^m\, e^{-\sum_{i=1}^n |z_i|^2/4l_B^2}$$

If η was a dynamical variable, as opposed to a parameter, this is just the original wavefunction with an extra electron at position η. But because η is not a dynamical variable, but instead a parameter, it's really a Laughlin wavefunction that describes a deficit of a single electron at position η. This means that m holes act like a deficit of a single electron, so a single quasi-hole is $1/m^{\rm th}$ of an electron. It should therefore carry charge $+e/m$.

We can make exactly the same argument in the context of the plasma analogy for the quasi-hole wavefunction (3.18). The resulting plasma potential energy has an extra term compared to (3.9),

$$U(z_i) = -m^2\sum_{i<j}\log\left(\frac{|z_i - z_j|}{l_B}\right) - m\sum_i \log\left(\frac{|z_i - \eta|}{l_B}\right) + \frac{m}{4l_B^2}\sum_{i=1}^N |z_i|^2$$

This extra term looks like an impurity in the plasma with charge 1. The particles in the plasma are expected to swarm around and screen this impurity. Each particle corresponds to a single electron, but has charge $q = -m$ in the plasma. The impurity carries $-1/m$ the charge of the electron. So the effective charge that's missing is $+1/m$; this is the charge of the quasi-hole.

The existence of fractional charge is very striking. We'll discuss this phenomenon more in the following section, but we'll postpone a direct derivation of fractional charge until Section 3.2.3 where we also discuss the related phenomenon of fractional statistics.

Quasi-Particles

There are also excitations of the quantum Hall fluid which carry charge $e^* = -e/m$, i.e. the same sign as the charge of an electron. These are *quasi-particles*.

It seems to be somewhat harder to write down quasi-particle eigenstates compared to quasi-hole eigenstates. To see the problem, note that we want to increase the density of electrons inside the Hall fluid and, hence, decrease the relative angular momentum of some pair of electrons. In the case of the quasi-hole, it was simple enough to *increase* the angular momentum: for example, for a hole at the origin we simply need to multiply the Laughlin wavefunction by the factor $\prod_i z_i$. But now that we want to decrease the angular momentum, we're not allowed divide by $\prod_i z_i$ as the resulting wavefunction is badly singular. Nor can we multiply by $\prod_i \bar{z}_i$ because, although this will decrease the angular momentum, the resulting wavefunction no longer sits in the lowest Landau level. Instead, a simple way to reduce the degree of a polynomial is to differentiate. This leads us to a candidate wavefunction for the quasi-particle,

$$\psi_{\text{particle}}(z,\eta) = \left[\prod_{i=1}^{N}\left(2\frac{\partial}{\partial z_i} - \bar{\eta}\right)\prod_{k<l}(z_k - z_l)^m\right] e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2} \tag{3.20}$$

Here the derivatives act only on the polynomial pre-factor; not on the exponential. The factor of $1/2$ in front of the position of the quasi-particle comes from a more careful analysis.

The quasi-particle wavefunction (3.20) is not quite as friendly as the quasi-hole wavefunction (3.18). For a start, the derivatives make it harder to work with and, for this reason, we will mostly derive results for quasi-holes in what follows. Further, the quasi-hole wavefunction (3.18) is an eigenstate of the toy Hamiltonian (3.15) (we'll see why shortly) while (3.20) is not. In fact, as far as I'm aware, the quasi-particle eigenstate of the toy Hamiltonain is not known.

Neutral Excitations

Before we proceed, we mention in passing that there are also neutral, collective excitations of the quantum Hall fluid in which the density and charge ripples in wave-like behaviour over large distances. These are similar to the phonon excitations in superfluids, except the energy cost does not vanish as the momentum $\hbar k \to 0$. The fact that these modes are gapped at $k = 0$ is the statement that the quantum Hall liquid is incompressible. In both cases, the energy-momentum dispersion relation exhibits a minimum at some finite wavevector k, referred to as a *roton* in superfluids and a *magneto-roton* in quantum Hall fluids. In both cases this is indicating the desire of the

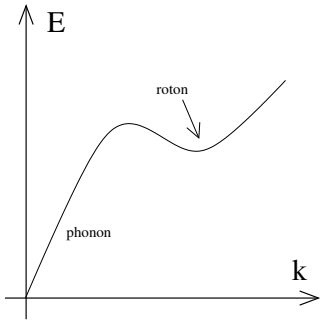


Figure 27: A cartoon of the dispersion relation in superfluids...

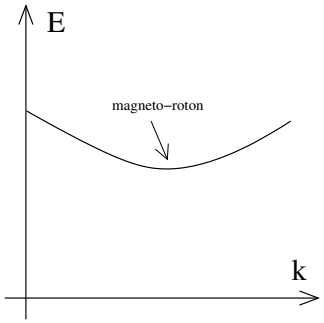


Figure 28: ...and for neutral excitations in quantum Hall fluids.

liquid to freeze to a solid – which, for the quantum Hall fluid is a Wigner crystal. In both cases, this desire is ultimately thwarted by quantum fluctuations.

In the quantum Hall fluid, the minimum occurs at momentum $k \sim 1/l_B$. In recent years, experiment has shown there is a rich structure underlying this. In particular, at other filling fractions (which we will discuss in Section 3.3) more than one minima is observed. We will not discuss these neutral excitations in these lectures.

3.2.1 Fractional Charge

The existence of an object which carries fractional electric charge is rather surprising. In this section, we'll explore some consequences.

Hall Conductivity Revisited

The most basic question we should ask of the Laughlin state is: does it reproduce the right Hall conductivity? To see that it does, we can repeat the Corbino disc argument of Section 2.2.2. As before, we introduce a flux $\Phi(t)$ into the centre of the ring which we slowly increase from zero to Φ_0. This induces a spectral flow so that when we reach $\Phi = \Phi_0$ we sit in a new eigenstate of the Hamiltonian in which the angular momentum of each electron increased by one. This is achieved by multiplying the wavefunction by the factor $\prod_i z_i$. We could even do this procedure in the case where both the inner circle and the inserted solenoid become vanishingly small. In this case, multiplying by $\prod_i z_i$ gives us precisely the quasi-hole wavefunction (3.18) with $\eta = 0$.

As an aside, note that we can also make the above argument above tells us that the quasi-hole wavefunction with $\eta = 0$ must be an eigenstate of the toy Hamiltonian (3.17), and indeed it is. (The wavefunction with $\eta \neq 0$ is also an eigenstate in the presence of the confining potential if we replace $\eta \to \eta e^{i\omega t}$, which tells us that the confining potential causes the quasi-hole to rotate).

We learn that as we increase Φ from zero to Φ_0, a particle of charge $-e/m$ is transferred from the inner to the outer ring. This means that a whole electron is transferred only when the flux is increased by $m\Phi_0$ units. The resultant Hall conductivity is

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\frac{1}{m}$$

as expected.

One can also ask how to reconcile the observed fractional Hall conductivity with the argument for integer quantisation based on Chern numbers when the Hall state is placed on a torus. This is slightly more subtle. It turns out that the ground state of the quantum Hall system on a torus is degenerate, hence violating one of the assumptions of the computation of the Chern number. We'll discuss this more in Section 3.2.4.

Measuring Fractional Electric Charge

It's worth pausing to describe in what sense the quasi-particles of the quantum Hall fluid genuinely carry fractional charge. First, we should state the obvious: we haven't violated any fundamental laws of physics here. If you isolate the quantum Hall fluid and measure the total charge you will always find an integer multiple of the electron charge.

Nonetheless, if you inject an electron (or hole) into the quantum Hall fluid, it will happily split into m seemingly independent quasi-particles (or quasi-holes). The states have a degeneracy labelled by the positions η_i of the quasi-objects. Moreover, these positions will respond to outside influences, such a confining potentials or applied electric fields, in the sense that the you can build solutions to the Schrödginer equation by endowing the positions with suitable time dependence $\eta_i(t)$. All of this means that the fractionally charged objects truly act as independent particles.

Indeed, the fractional charge can be seen experimentally in *shot noise* experiments. This is a randomly fluctuating current, where the fluctuations can be traced to the discrete nature of the underlying charge carriers. This allows a direct measurement[15]. of the charge carriers which, for the $\nu = 1/3$ state, were shown to indeed carry charge $e^\star = e/3$

[15]The experiment was first described in R. de-Picciotto, M. Reznikov, M. Heiblum, V. Umansky, G. Bunin, and D. Mahalu, "*Direct observation of a fractional charge*", Nature 389, 162 (1997). cond-mat/9707289.

3.2.2 Introducing Anyons

We're taught as undergrads that quantum particles fall into two categories: bosons and fermions. However, if particles are restricted to move in a two-dimensional plane then there is a loophole to the usual argument and, as we now explain, much more interesting things can happen[16].

Let's first recall the usual argument that tells us we should restrict to boson and fermions. We take two identical particles described by the wavefunction $\psi(\mathbf{r}_1, \mathbf{r}_2)$. Since the particles are identical, all probabilities must be the same if the particles are exchanged. This tells us that $|\psi(\mathbf{r}_1, \mathbf{r}_2)|^2 = |\psi(\mathbf{r}_2, \mathbf{r}_1)|^2$ so that, upon exchange, the wavefunctions differ by at most a phase

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\pi\alpha}\psi(\mathbf{r}_2, \mathbf{r}_1) \tag{3.21}$$

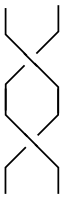
Now suppose that we exchange again. Performing two exchanges is equivalent to a rotation, so should take us back to where we started. This gives the condition

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{2i\pi\alpha}\psi(\mathbf{r}_1, \mathbf{r}_2) \qquad \Rightarrow \qquad e^{2\pi i\alpha} = 1$$

This gives the two familiar possibilities of bosons ($\alpha = 0$) or fermions ($\alpha = 1$).

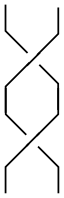
So what's the loophole in the above argument? The weak point is the statement that when we rotate two particles by 360° we should get back to where we came from. Why should this be true? The answer lies in thinking about the topology of the worldlines particles make in spacetime.

In $d = 3$ spatial dimensions (and, if you're into string theory, higher), the path that the pair of particles take in spacetime can always be continuously connected to the situation where the particles don't move at all. This is the reason the resulting state should be the same as the one before the exchange. But in $d = 2$ spatial dimensions, this is not the case: the worldlines of particles now wind around each other. When particles are exchanged in an anti-clockwise direction, like this



[16]This possibility was first pointed out by Jon Magne Leinaas and Jan Myrheim, "*On the Theory of Identical Particles*", Il Nuovo Cimento B37, 1-23 (1977). This was subsequently rediscovered by Frank Wilczek in "*Quantum Mechanics of Fractional-Spin Particles*", Phys. Rev. Lett. 49 (14) 957 (1982).

the worldlines get tangled. They can't be smoothly continued into the worldlines of particles which are exchanged clockwise, like this:



Each winding defines a different topological sector. The essence of the loophole is that, after a rotation in the two-dimensions, the wavefunction may retain a memory of the path it took through the phase. This means that may have any phase α in (3.21). In fact, we need to be more precise: we will say that after an *anti-clockwise* exchange, the wavefunction is

$$\psi(\mathbf{r}_1, \mathbf{r}_2) = e^{i\pi\alpha}\psi(\mathbf{r}_2, \mathbf{r}_1) \tag{3.22}$$

After a clockwise exchange, the phase must be $e^{-i\pi\alpha}$. Particles with $\alpha \neq 0, 1$ are referred to as *anyons*. This whole subject usually goes by the name of *quantum statistics* or *fractional statistics*. But it has less to do with statistics and more to do with topology.

The Braid Group

Mathematically, what's going on is that in dimensions $d \geq 3$, the exchange of particles must be described by a representation of the permutation group. But, in $d = 2$ dimensions, exchanges are described a representation of the *braid group*.

Suppose that we have n identical particles sitting along a line. We'll order them $1, 2, 3, \ldots, n$. The game is that of a street-magician: we shuffle the order of the particles. The image that their worldlines make in spacetime is called a braid. We'll only distinguish braids by their topological class, which means that two braids are considered the same if we can smoothly change one into the other without the worldlines crossing. All such braidings form an infinite group which we call B_n

We can generate all elements of the braid group from a simple set of operations, $R_1, \ldots, R_{n-1}$ where R_i exchanges the i^{th} and $(i+1)^{\text{th}}$ particle in an anti-clockwise direction. The defining relations obeyed by these generators are

$$R_i R_j = R_j R_i \quad |i - j| > 2$$

together with the *Yang-Baxter relation*,

$$R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \qquad i = 1, \ldots, n-1$$

This latter relation is most easily seen by drawing the two associated braids and noting that one can be smoothly deformed into the other.

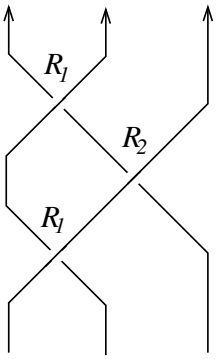


Figure 29: The left hand-side of the Yang-Baxter equation...

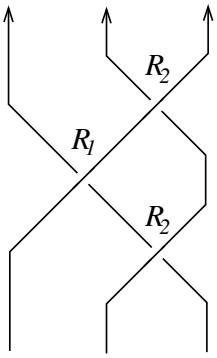


Figure 30: ...is topologically equivalent to the right-hand side.

In quantum mechanics, exchanges of particles act as unitary operators on the Hilbert space. These will form representations of the braid group. The kind of anyons that we described above form a one-dimensional representation of the braid group in which each exchange just gives a phase: $R_i = e^{i\pi\alpha_i}$. The Yang-Baxter equation then requires $e^{i\pi\alpha_i} = e^{i\pi\alpha_{i+1}}$ which simply tells us that all identical particles must have the same phase.

One dimensional representation of the braid group are usually referred to as *Abelian anyons*. As we'll show below, these are the kind of anyons relevant for the Laughlin states. However, there are also more exotic, higher-dimensional representations of the braid group. These are called *non-Abelian anyons*. We will discuss some examples in Section 4.

3.2.3 Fractional Statistics

We will now compute the quantum statistics of quasi-holes in the $\nu = 1/m$ Laughlin state. In passing, we will also provide a more sophisticated argument for the fractional charge of the quasi-hole. Both computations involve the Berry phase that arises as quasi-holes move[17].

We consider a state of M quasi-holes which we denote as $|\eta_1, \ldots, \eta_M\rangle$. The wavefunction is (3.19)

$$\langle z, \bar{z}|\eta_1, \ldots, \eta_M\rangle = \prod_{j=1}^{M}\prod_{i=1}^{N}(z_i - \eta_j)\prod_{k<l}(z_k - z_l)^m\, e^{-\sum_{i=1}^{n}|z_i|^2/4l_B^2}$$

[17]The structure of this calculation was first described in Daniel Arovas, John Schrieffer and Frank Wilczek, "*Fractional statistics and the quantum Hall effect*", Phys. Rev. Lett. 53, 772 (1984), although they missed the importance of working with normalised wavefunctions. This was subsequently clarified by M. Stone. in the collection of reprints he edited called, simply, "*The Quantum Hall Effect*".

However, whenever we compute the Berry phase, we should work with the normalised states. We'll call this state $|\psi\rangle$, defined by

$$|\psi\rangle = \frac{1}{\sqrt{Z}}|\eta_1, \ldots, \eta_M\rangle$$

where the normalisation factor is defined as $Z = \langle\eta_1, \ldots, \eta_M|\eta_1, \ldots, \eta_M\rangle$, which reads

$$Z = \int \prod d^2 z_i \ \exp\left(\sum_{i,j} \log|z_i - \eta_j|^2 + m\sum_{k,l}\log|z_k - z_l|^2 - \frac{1}{2l_B^2}\sum_i |z_i|^2\right) \quad (3.23)$$

This is the object which plays the role of the partition function in the plasma analogy, now in the presence of impurities localised at η_i.

The holomorphic Berry connection is

$$\mathcal{A}_\eta(\eta, \bar\eta) = -i\langle\psi|\frac{\partial}{\partial\eta}|\psi\rangle = \frac{i}{2Z}\frac{\partial Z}{\partial\eta} - \frac{i}{Z}\langle\eta|\frac{\partial}{\partial\eta}|\eta\rangle$$

But because $|\eta\rangle$ is holomorphic, and correspondingly $\langle\eta|$ is anti-holomorphic, we have $\frac{\partial Z}{\partial\eta} = \frac{\partial}{\partial\eta}\langle\eta|\eta\rangle = \langle\eta|\frac{\partial}{\partial\eta}|\eta\rangle$. So we can write

$$\mathcal{A}_\eta(\eta, \bar\eta) = -\frac{i}{2}\frac{\partial \log Z}{\partial\eta}$$

Meanwhile, the anti-holomorphic Berry connection is

$$\mathcal{A}_{\bar\eta}(\eta, \bar\eta) = -i\langle\psi|\frac{\partial}{\partial\bar\eta}|\psi\rangle = +\frac{i}{2}\frac{\partial \log Z}{\partial\bar\eta}$$

So our task in both cases is to compute the derivative of the partition function (3.23). This is difficult to do exactly. Instead, we will invoke our intuition for the behaviour of plasmas.

Here's the basic idea. In the plasma analogy, the presence of the hole acts like a charged impurity. In the presence of such an impurity, the key physics is called *screening*[18]. This is the phenomenon in which the mobile charges – with positions z_i – rearrange themselves to cluster around the impurity so that its effects cannot be noticed when you're suitably far away. More mathematically, the electric potential due to the impurity is modified by an exponential fall-off $e^{-r/\lambda}$ where λ is called the *Debye screening length* and is proportional to $\sqrt{T}$. Note that, in order for us to use this argument, it's crucial that the artificial temperature (3.8) is high enough that the plasma lies in the fluid phase and efficient screening can occur.

[18]You can read about screening in the final section of the lecture notes on *Electromagnetism*.

Whenever such screening occurs, the impurities are effectively hidden at distances much greater than λ. This means that the free energy of the plasma is independent of the positions of the impurities, at least up to exponentially small corrections. This free energy is, of course, proportional to $\log Z$ which is the thing we want to differentiate. However, there are two ingredients missing: the first is the energy cost between the impurities and the constant background charge; the second is the Coulomb energy between the different impurities. The correct potential energy for the plasma with M impurities should therefore be

$$U(z_k;\eta_i) = -m^2 \sum_{k<l} \log\left(\frac{|z_k - z_l|}{l_B}\right) - m\sum_{k,i} \log\left(\frac{|z_i - \eta_i|}{l_B}\right) - \sum_{i<j} \log\left(\frac{|\eta_i - \eta_j|}{l_B}\right) + \frac{m}{4l_B^2}\sum_{k=1}^{N}|z_k|^2 + \frac{1}{4l_B^2}\sum_{i=1}^{M}|\eta_i|^2 \tag{3.24}$$

The corrected plasma partition function is then

$$\int \prod d^2 z_i \; e^{-\beta U(z_i;\eta)} = \exp\left(-\frac{1}{m}\sum_{i<j}\log|\eta_i - \eta_j|^2 + \frac{1}{2ml_B^2}\sum_i |\eta_i|^2\right) Z$$

As long as the distances between impurities $|\eta_i - \eta_j|$ are greater than the Debye length λ, the screening argument tells us that this expression should be independent of the positions η_i for high enough temperature. In particular, as we described previously, the temperature $\beta = 2/m$ relevant for the plasma analogy is high enough for screening as long as $m \lesssim 70$. This means that we must have

$$Z = C\exp\left(\frac{1}{m}\sum_{i<j}\log|\eta_i - \eta_j|^2 - \frac{1}{2ml_B^2}\sum_i |\eta_i|^2\right)$$

for some constant C which does not depend on η_i. This gives some idea of the power of the plasma analogy. It looks nigh on impossible to perform the integrals in (3.23) directly; yet by invoking some intuition about screening, we are able to write down the answer, at least in some region of parameters.

The Berry connections over the configuration space of M quasi-holes are then simple to calculate: they are

$$\mathcal{A}_{\eta_i} = -\frac{i}{2m}\sum_{j\neq i}\frac{1}{\eta_i - \eta_j} + \frac{i\bar{\eta}_i}{4ml_B^2} \tag{3.25}$$

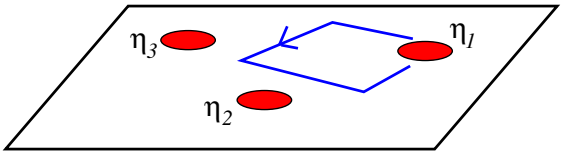


Figure 31: The path taken to compute the fractional charge of the quasi-hole...

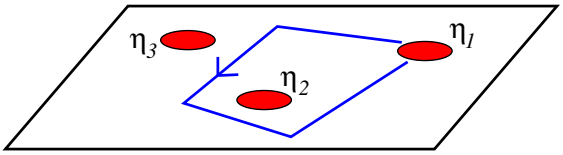


Figure 32: ...and the path to compute the fractional statistics.

and

$$\mathcal{A}_{\bar{\eta}_i} = +\frac{i}{2m}\sum_{j\neq i}\frac{1}{\bar{\eta}_i - \bar{\eta}_j} - \frac{i\eta_i}{4ml_B^2} \tag{3.26}$$

where we stress that these expressions only hold as long as the quasi-holes do not get too close to each other where the approximation of complete screening breaks down. We can now use these Berry connections to compute both the charge and statistics of the quasi-hole.

Fractional Charge

Let's start by computing the charge of the anyon. The basic idea is simple. We pick one of the quasi-holes — say $\eta_1 \equiv \eta$ — and move it on a closed path C. For now we choose a path which does not enclose any of the other anyons. This ensures that only the second term in the Berry phase contributes,

$$\mathcal{A}_\eta = \frac{i\bar{\eta}}{4ml_B^2} \quad \text{and} \quad \mathcal{A}_{\bar{\eta}} = -\frac{i\eta}{4ml_B^2}$$

After traversing the path C, the quasi-hole will return with a phase shift of $e^{i\gamma}$, given by the Berry phase

$$e^{i\gamma} = \exp\left(-i\oint_C \mathcal{A}_\eta d\eta + \mathcal{A}_{\bar{\eta}} d\bar{\eta}\right) \tag{3.27}$$

This gives the Berry phase

$$\gamma = \frac{e\Phi}{m\hbar} \tag{3.28}$$

where Φ is the total magnetic flux enclosed by the path C. But there's a nice interpretation of this result: it's simply the Aharonov-Bohm phase picked up by the particle. As described in Section 1.5.3, a particle of charge $e^\star$ will pick up phase $\gamma = e^\star\Phi/\hbar$. Comparing to (3.28), we learn that the charge of the particle is indeed

$$e^\star = \frac{e}{m}$$

as promised.

Fractional Statistics

To compute the statistics, we again take a particular quasi-hole — say η_1 — on a journey, this time on a path C which encloses one other quasi-hole, which we'll take to be η_2. The phase is once again given by (3.27) where, this time, both terms in the expressions (3.25) and (3.26) for $\mathcal{A}_\eta$ and $\mathcal{A}_{\bar{\eta}}$ contribute. The second term once again gives the Aharonov-Bohm phase; the first term tells us about the statistics. It is

$$e^{i\gamma} = \exp\left(-\frac{1}{2m}\oint_C \frac{d\eta_1}{\eta_1-\eta_2} + \text{h.c.}\right) = e^{2\pi i/m}$$

This is the phase that arises from one quasi-hole encircling the other. But the quantum statistics comes from *exchanging* two objects, which can be thought of as a rotating by 180° rather than 360°. This means that, in the notation of (3.22), the phase above is

$$e^{2\pi i\alpha} = e^{2\pi i/m} \quad \Rightarrow \quad \alpha = \frac{1}{m} \tag{3.29}$$

Note that for a fully filled Landau level, with $m = 1$, the quasi-holes are fermions. (They are, of course, actual holes). But for a fractional quantum Hall state, the quasi-holes are anyons.

Suppose now that we put n quasi-holes together and consider this as a single object. What are its statistics? If we exchange two such objects, then each quasi-hole in the first bunch gets exchanged with each quasi-hole in the second bunch. The net result is that the statistical parameter for n quasi-holes is $\alpha = n^2/m$ (recall that the parameter α is defined mod 2). Note that α does not grow linearly with n. As a check, suppose that we put m quasi-holes together to reform the original particle that underlies the Hall fluid. We get $\alpha = m^2/m = m$ which is a boson for m even and a fermion for m odd.

There's a particular case of this which is worth highlighting. The quasi-particles in the $m = 2$ bosonic Hall state have statistical parameter $\alpha = 1/2$. They are half-way between bosons and fermions and sometimes referred to as *semions*. Yet two semions do not make a fermion; they make a boson.

More generally, it's tempting to use this observation to argue that an electron can only ever split into an odd number of anyons. This argument runs as follow: if an electron were to split into an even number of constituents n, each with statistical parameter α, then putting these back together again would result in a particle with statistical parameter $n^2\alpha$. The argument sounds compelling. However, as we will see in Section 4, there is a loop hole!

While the fractional charge of quasi-holes has been measured experimentally, a direct detection of their fractional statistics is more challenging. There have been a number of proposed (and performed) experiments using interferometry to demonstrate but their conclusions remain open to interpretation.

A Slightly Different Viewpoint

There is a slightly different way of presenting the calculation. It will offer nothing new here, but often appears in the literature as it proves useful when discussing more complicated examples. The idea is that we consider a wavefunction that already has the interesting η dependence built in. So, instead of (3.19), we work with

$$\psi = \prod_{a<b}(\eta_a - \eta_b)^{1/m} \prod_{a,i}^{N}(z_i - \eta_a) \prod_{k<l}(z_k - z_l)^m \, e^{-\sum_i |z_i|^2/4l_B^2 - \sum_a |\eta_a|^2/4ml_B^2} \tag{3.30}$$

This wavefunction is cooked up so that the associated probability distribution is given precisely by the partition function with energy (3.24) and hence has no dependence on η and $\bar{\eta}$. This means that the Berry connection for this wavefunction has only the second terms in (3.25) and (3.26), corresponding to the Aharonov-Bohm effect due to the background magnetic field. The term in the Berry connection that was responsible for fractional statistics is absent. But this doesn't mean that the physics has changed. Instead, this phase is manifest in the form of the wavefunction itself, which is no longer single-valued in η_a. Indeed, if η_1 encircles a neighbouring point η_2, the wavefunction pick up a phase $e^{2\pi i/m}$, so exchanging two quasi-holes gives the phase $e^{i\pi/m}$.

Of course, this approach doesn't alleviate the need to determine the Berry phase arising from the exchange. You still need to compute it to check that it is indeed zero.

3.2.4 Ground State Degeneracy and Topological Order

In this section we describe a remarkable property of the fractional quantum Hall states which only becomes apparent when you place them on a compact manifold: the number of ground states depends on the topology of the manifold. As we now explain, this is intimately related to the existence of anyonic particles.

Consider the following process on a torus. We create from the vacuum a quasi-particle – quasi-hole pair. We then separate this pair, taking them around one of the two different cycles of the torus as shown in the figure, before letting them annihilate again. We'll call the operator that implements this process T_1 for the first cycle and T_2 for the second.

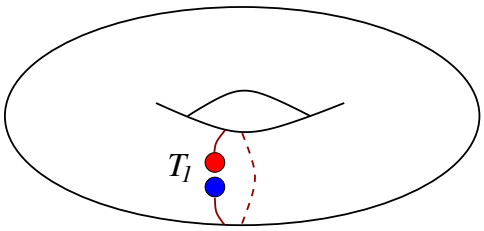


Figure 33: Taking a quasi-hole (red) and quasi-particle (blue) around one cycle of the torus

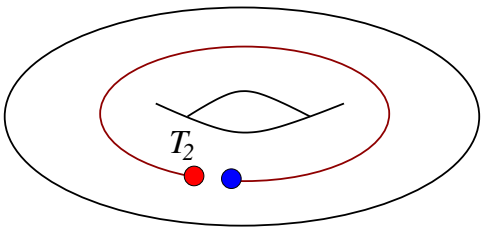


Figure 34: ...or around the other.

Now suppose we take the particles around one cycle and then around the other. Because the particles are anyons, the order in which we do this matters: there is a topological difference between the paths taken. Indeed, you can convince yourself that $T_1T_2T_1^{-1}T_2^{-1}$ is equivalent to taking one anyon around another: the worldlines have linking number one. This means that the T_i must obey the algebra

$$T_1T_2 = e^{2\pi i/m}\, T_2T_1 \tag{3.31}$$

But such an algebra of operators can't be realised on a single vacuum state. This immediately tells us that the ground state must be degenerate. The smallest representation of (3.31) has dimension m, with the action

$$T_1|n\rangle = e^{2\pi ni/m}|n\rangle$$
$$T_2|n\rangle = |n+1\rangle$$

The generalisation of this argument to a genus-g Riemann surface tells us that the ground state must have degeneracy m^g. Notice that we don't have to say anything about the shape or sizes of these manifolds. The number of ground states depends only on the topology!

It is also possible to explicitly construct the analog of the Laughlin states on a torus in terms of Jacobi theta functions and see that there are indeed m such states.

Before we proceed, we note that this resolves a puzzle. In Section 2.2.4, we described a topological approach to the integer quantum Hall effect which is valid when space is a torus. With a few, very mild, assumptions, we showed that the Hall conductivity is equal to a Chern number and must, therefore, be quantised. In particular, this calculation made no assumption that the electrons were non-interacting: it holds equally well for strongly interacting many-body systems. However, one of the seemingly mild assumptions was that the ground state was non-degenerate. As we've seen, this is not true for fractional quantum Hall states, a fact which explains how these states avoid having integer Hall conductivity.

Topological Order

We've seen in this section that the Laughlin states have a number of very special properties. One could ask: how can we characterise these states? This is an old and venerable question in condensed matter physics and, for most systems, has an answer provided by Landau. In Landau's framework, different states of matter are characterised by their symmetries, both those that are preserved by the ground state and those that are broken. This is described using order parameters of the kind that we met in the lectures on *Statistical Physics* when discussing phase transitions.

However, the quantum Hall fluids fall outside of this paradigm. There is no symmetry or local order parameter that distinguishes quantum Hall states. It turns out that there is a non-local order parameter, usually called "off-diagonal long-range order" and this can be used to motivate a Ginzburg-Landau-like description. We will describe this in Section 5.3.2 but, as we will see, it is not without its pitfalls.

Instead, Wen[19] suggested that we should view quantum Hall fluids as a new type of matter, characterised by *topological order*. The essence of the proposal is that quantum states can be characterised their ground state degeneracy and the way in which these states transform among themselves under operations like (3.31).

3.3 Other Filling Fractions

So far, we have only described the quantum Hall states at filling fraction $\nu = 1/m$. Clearly there are many more states that are not governed by the Laughlin wavefunction. As we now show, we can understand many of these by variants of the ideas above.

A Notational Convention

Before we proceed, let's quickly introduce some new notation. All wavefunctions in the lowest Landau level come with a common exponential factor. It gets tiresome writing it all the time, so define

$$\psi(z,\bar{z}) \sim \tilde{\psi}(z) e^{-\sum_{i=1}^{n} |z_i|^2/4l_B^2}$$

where $\tilde{\psi}(z)$ is a holomorphic function. In what follows we will often just write $\tilde{\psi}(z)$. Be warned that many texts drop the exponential factor in the wavefunctions but don't give the resulting object a different name.

[19] The original paper is Xiao-Gang Wen, "*Topological Orders in Rigid States*", Int. J. Mod. Phys. B4, 239 (1990), available at Xiao-Gang's website.

3.3.1 The Hierarchy

We saw in Section 3.2.1 how one can induce quasi-hole (or quasi-particle) states by introducing a local excess (or deficit) of magnetic field through a solenoid. We could also ask what happens if we change the magnetic field in a uniform manner so that the system as a whole moves away from $\nu = 1/m$ filling. For definiteness, suppose that we increase B so that the filling fraction decreases. It seems plausible that for B close to the initial Laughlin state, the new ground state of the system will contain some density of quasi-holes, arranged in some, perhaps complicated, configuration. The key idea of this section is that these quasi-holes might themselves form a quantum Hall state. Let's see how this would work.

We know that Laughlin states take the form

$$\tilde{\psi} \sim \prod_{i<j} (z_i - z_j)^m$$

where m is odd for fermions and even for bosons. What would a Laughlin state look like for anyons with positions η_i and statistical parameter α? To have the right statistics, the wavefunctions must take the form

$$\tilde{\psi} \sim \prod_{i<j}^{N} (\eta_i - \eta_j)^{2p+\alpha}$$

with p a positive integer. As we've seen, above the $\nu = 1/m$ state, quasi-holes have statistics $\alpha = 1/m$ while quasi-particles have statistics $\alpha = -1/m$. It's simple to repeat our previous counting of the filling fraction, although now we need to be more careful about what we're counting. The maximum angular momentum of a given quasi-excitation is $N(2p \pm \frac{1}{m})$ so the area of the droplet is $A \approx 2\pi(2p \pm \frac{1}{m})N(ml_B^2)$ where the usual magnetic length $l_B^2 = \hbar/eB$ is now replaced by ml_B^2 because the charge of the quasi-excitations is $q = \pm e/m$. The number of *electron* states in a full Landau level is AB/Φ_0 and each can be thought of as made of m quasi-things. So the total number of quasi-thing states in a full Landau level is $mAB/\Phi_0 = (2p \pm \frac{1}{m})m^2N$.

The upshot of this is that the quasi-holes or quasi-particles give a contribution to the filling of electron states

$$\nu_{\text{quasi}} = \mp \frac{1}{2pm^2 \pm m}$$

where the overall sign is negative for holes and positive for particles. Adding this to the filling fraction of the original $\nu = 1/m$ state, we have

$$\nu = \frac{1}{m} \mp \frac{1}{2pm^2 \pm m} = \frac{1}{m \pm \frac{1}{2p}} \tag{3.32}$$

Note that the filling fraction is decreased by quasi-holes and increased by quasi-particles.

Let's look at some simple examples. We start with the $\nu = 1/3$ state. The $p = 1$ state for quasi-particles then gives $\nu = 2/5$ which is one of the more prominent Hall plateaux. The $p = 1$ state for quasi-hole gives $\nu = 2/7$ which has also been observed; while not particularly prominent, it's harder to see Hall states at these lower filling fractions.

Now we can go further. The quasi-objects in this new state can also form quantum Hall states. And so on. The resulting fillings are given by the continuous fractions

$$\nu = \cfrac{1}{m \pm \cfrac{1}{2p_1 \pm \cfrac{1}{2p_2 \pm \cdots}}} \tag{3.33}$$

For example, building on the Hall state $\nu = 1/3$, the set of continuous fractions for quasi-particles with $p_i = 1$ leads to the sequence $\nu = 2/5$ (which is the fraction (3.32)), followed by $\nu = 3/7$, $4/9$, $5/11$ and $6/13$. This is precisely the sequence of Hall plateaux shown in the data presented at the beginning of this chapter.

3.3.2 Composite Fermions

We now look at an alternative way to think about the hierarchy known as *composite fermions*[20]. Although the starting point seems to be logically different from the ideas above, we will see the same filling fractions emerging. Moreover, this approach will allow us to go further ending, ultimately, in Section 3.3.3 with a striking prediction for what happens at filling fraction $\nu = 1/2$.

First, some motivation for what follows. It's often the case that when quantum systems become strongly coupled, the right degrees of freedom to describe the physics are not those that we started with. Instead new, weakly coupled degrees of freedom may emerge. Indeed, we've already seen an example of this in the quantum Hall effect, where we start with electrons but end up with fractionally charged particles.

[20]This concept was first introduced by Jainendra Jain in the paper "*Composite-Fermion Approach to the Fractional Quantum Hall Effect*", Phys. Rev. Lett. 63 2 (1989). It is reviewed in some detail in his book called, appropriately, "*Composite Fermions*". A clear discussion can also be found in the review "*Theory of the Half Filled Landau Level*" by Nick Read, cond-mat/9501090.

The idea of this section is to try to find some new degrees of freedom — these are the "composite fermions". However, for the most part these won't be the degrees of freedom that are observed in the system. Instead, they play a role in the intermediate stages of the calculations. (There is an important exception to this statement which is the case of the half-filled Landau level, described in Section 3.3.3, where the observed excitations of the system are the composite fermions.) Usually it is difficult to identify the emergent degrees of freedom, and it's no different here. We won't be able to rigorously derive the composite fermion picture. Instead, we'll give some intuitive and, in parts, hand-waving arguments that lead us to a cartoon description of the physics. But the resulting cartoon is impressively accurate. It gives ansatze for wavefunctions which are in good agreement with the numerical studies and it provides a useful and unified way to think about different classes of quantum Hall states.

We start by introducing the idea of a *vortex*. Usually a vortex is a winding in some complex order parameter. Here, instead, a vortex will mean a winding in the wavefunction itself. Ultimately we will be interested in vortices in the Laughlin wavefunction, but to understand the key physics it's simplest to revisit the quasi-hole whose wavefunction includes the factor

$$\prod_i (z_i - \eta)$$

Clearly the wavefunction now has a zero at the position η. This does two things. First, it depletes the charge there. This, of course, is what gives the quasi-hole its fractional charge e/m. But because the lowest Landau level wavefunction is holomorphic, there is also fixed angular dependence: the phase of the wavefunction winds once as the position of any particle moves around η. This is the vortex.

The winding of the wavefunction is really responsible for the Berry phase calculations we did in Section 3.2.3 to determine the fractional charge and statistics of the quasi-hole. Here's a quick and dirty explanation. The phase of the wavefunction changes by 2π as a particle moves around the quasi-hole. Which means that it should also change by 2π when the quasi-hole moves around the particle. So if we drag the quasi-hole around $N = \nu\Phi/\Phi_0$ particles, then the phase changes by $\gamma = 2\pi N = \nu e\Phi/\hbar$. This is precisely the result (3.28) that we derived earlier. Meanwhile, if we drag one quasi-hole around a region in which there is another quasi-hole, the charge inside will be depleted by e/m, so the effective number of particles inside is now $N = \nu\Phi/\Phi_0 - 1/m$. This gives an extra contribution to the phase $\gamma = -2\pi/m$ which we associate the statistics of the quasi-holes: $\gamma = 2\pi\alpha = 2\pi/m$ so $\alpha = 1/m$, reproducing our earlier result (3.29). We stress that all of these results really needed only the vortex nature of the quasi-hole.

Now let's turn to the Laughlin wavefunction itself

$$\tilde{\psi}_m(z) \sim \prod_{i<j}(z_i - z_j)^m$$

For now we focus on m odd so that the wavefunction is anti-symmetric and we're dealing with a Hall state of fermions. One striking feature is that the wavefunction has a zero of order m as two electrons approach. This means that each particle can be thought of as m vortices. Of course, one of these zeros was needed by the Pauli exclusion principle. Moreover, we needed m zeros per particle to get the filling fraction right. But nothing forced us to have the other $m-1$ zeros sitting at exactly the same place. This is something special about the Laughlin wavefunction.

Motivated by this observation, we define a *composite fermion* to be an electron (which gives rise to one vortex due to anti-symmetry) bound to $m-1$ further vortices. The whole thing is a fermion when m is odd. You'll sometimes hear composite fermions described as electrons attached to flux. We'll describe this picture in the language of Chern-Simons theory in Section 5 but it's not particular useful in the present context. In particular, it's important to note that the composite fermions don't carry real magnetic flux with them. This remains uniform. Instead, as we will see later, they carry a different, emergent flux.

Let's try to treat this as an object in its own right and see what behaviour we find. Consider placing some density $n = \nu B/\Phi_0$ of electrons in a magnetic field and subsequently attaching these vortices to make composite fermions. We will first show that these composite fermions experience both a different magnetic field $B^\star$ and different filling fraction $\nu^\star$ than the electrons. To see this, we repeat our Berry phase argument where we move the composite fermion along a path encircling an area A. The resulting Berry phase has two contributions,

$$\gamma = 2\pi\left(\frac{AB}{\Phi_0} - (m-1)nA\right) \tag{3.34}$$

with n the density of electrons. The first term is the usual Aharonov-Bohm phase due to the total flux inside the electron path. The second term is the contribution from the electron encircling the vortices: there are $m-1$ such vortices attached to each of the ρA electrons.

When we discussed quasi-holes, we also found a different Aharonov-Bohm phase. In that context, we interpreted this as a different charge of quasi-particles. In the present context, one usually interprets the result (3.34) in a different (although ultimately

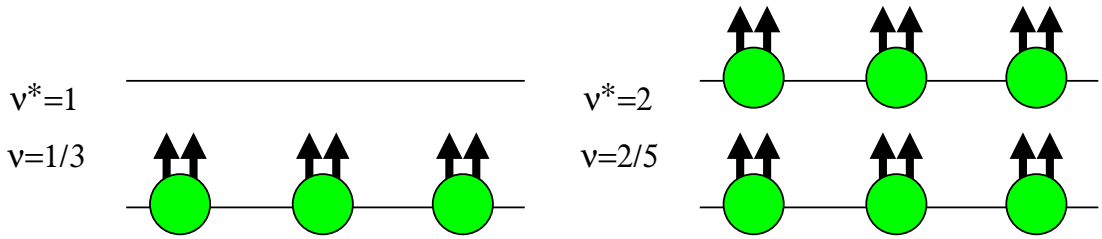


Figure 35: The composite fermion picture describes a hierarchy of plateaux around, starting with $\nu = 1/3$, in terms of the integer quantum Hall effect for electrons bound to two vortices.

equivalent) way: we say that the composite fermions experience a different magnetic field which we call $B^\star$. The Aharonov-Bohm phase should then be

$$\gamma = \frac{2\pi A B^\star}{\Phi_0} \quad \Rightarrow \quad B^\star = B - (m-1)n\Phi_0 \tag{3.35}$$

Because there is one electron per composite fermion, the density is the same. But because the magnetic fields experienced by electrons and composite fermions differ, the filling fractions must also differ: we must have $n = \nu^\star B^\star/\Phi_0 = \nu B/\Phi_0$. This gives

$$\nu = \frac{\nu^\star}{1 + (m-1)\nu^\star} \tag{3.36}$$

This is an interesting equation! Suppose that we take the composite fermions to completely fill their lowest Landau level, so that $\nu^\star = 1$. Then we have

$$\nu^\star = 1 \quad \Rightarrow \quad \nu = \frac{1}{m}$$

In other words, the fractional quantum Hall effect can be thought of as an integer quantum Hall effect for composite fermions. That's very cute! Indeed, we can even see some hint of this in the Laughlin wavefunction itself which we can trivially rewrite as

$$\tilde{\psi}_m(z) \sim \prod_{i<j}(z_i - z_j)^{m-1} \prod_{k<l}(z_k - z_l) \tag{3.37}$$

The second term in this decomposition is simply the wavefunction for the fully-filled lowest Landau level. We're going to think of the first term as attaching $m-1$ vortices to each position z_i to form the composite fermion.

So far we've said a lot of words, but we haven't actually derived anything new from this perspective. But we can extract much more from (3.36). Suppose that we fill the

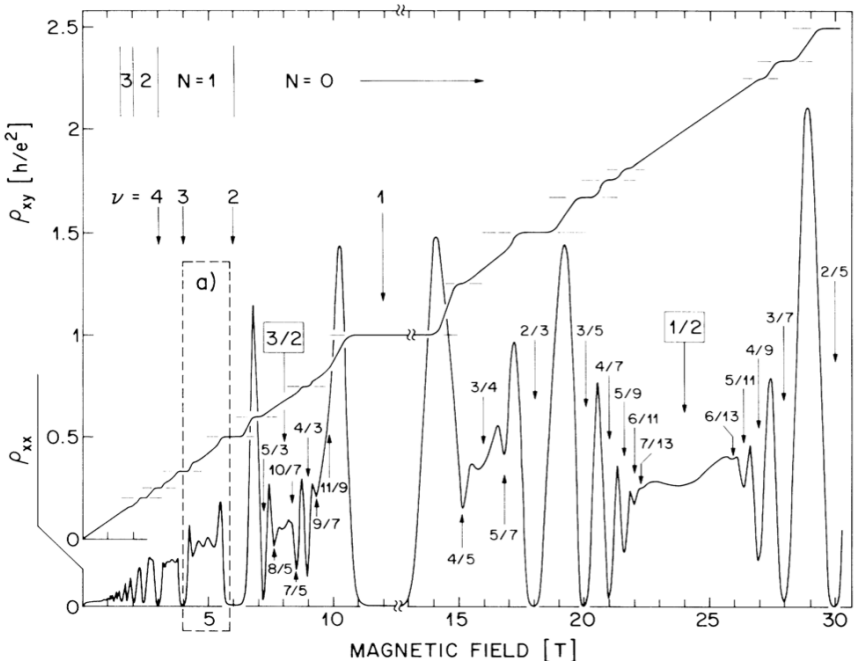


Figure 36: The fractional Hall plateaux....again

first $\nu^\star$ Landau levels of to get an integer quantum Hall effect for composite fermions with $\nu^\star > 1$. (The Landau levels for composite fermions are sometimes referred to Λ levels.) Then we find filling fractions that are different from the Laughlin states. For example, if we pick $m = 3$, then the sequence of states arising from (3.36) is $\nu = 1/3,\ 2/5,\ 3/7,\ 4/9, \ldots$. These is the same sequence that we saw in the hierarchy construction and is clearly visible in the data shown in the figure. Inspired by the form of (3.37), we will write down a guess for the wavefunction, usually referred to as *Jain states*,

$$\tilde{\psi}_\nu(z) = \mathcal{P}_{LLL}\left[\prod_{i<j}(z_i - z_j)^{m-1}\,\Psi_{\nu^\star}(z,\bar{z})\right] \tag{3.38}$$

Here $\Psi_{\nu^\star}$ is the wavefunction for $\nu^\star \in \mathbf{Z}$ fully-filled Landau levels while the $\prod(z_i - z_j)^{m-1}$ factor attaches the $(m-1)$ vortices to each electron. The wavefunction $\Psi_{\nu^\star}$ can be easily constructed by a Slater determinant of the form (3.5) except that, this time, we run into a problem. The electrons have filling fraction $\nu < 1$ and so are supposed to lie in the lowest Landau level. Meanwhile, the integer quantum Hall states $\Psi_{\nu^\star}$ are obviously not lowest Landau level wavefunctions: they depend on $\bar{z}_i$ as well as z_i. This is what the mysterious symbol $\mathcal{P}_{LLL}$ is doing in the equation (3.38): it means "project to the lowest Landau level".

We should define operationally what $\mathcal{P}_{LLL}$ actually does in this equation. It does *not* mean "take the state in [...] and project onto the component of the wavefunction which sits in the lowest Landau level". This is because there is no part of the wavefunction in [...] which sits in the lowest Landau level! Instead $\mathcal{P}_{LLL}$ means "take the state in [...] and artificially massage it so that it sits in the lowest Landau level". This is a brutal operation and there is no unique way to do it. The most straightforward is to move all factors of $\bar{z}_i$ in [...] to the left. We then make the substitution

$$\bar{z}_i \to 2l_B^2 \frac{\partial}{\partial z_i} \tag{3.39}$$

Note that this is the same kind of substitution we made in constructing the quasi-particle wavefunction (3.20). For a small number of particles ($N \approx 10$ or so) one can compute numerically the exact wavefunctions in different filling fractions: the wavefunctions (3.38) built using the procedure described above have an overlap of around 99% or so.

Note that it's also possible to have $B^\star < 0$. In this case, we have $\rho = -\nu^\star B^\star/\Phi_0$ and the relationship (3.36) becomes

$$\nu = \frac{\nu^\star}{(m-1)\nu^\star - 1}$$

Then filling successive Landau levels $\nu^\star \in \mathbf{Z}$ gives the sequence $\nu = 1,\ 2/3,\ 3/5,\ 4/7,\ 5/9, \ldots$ which we again see as the prominent sequence of fractions sitting to the left of $\nu = 1/2$ in the data.

We can also use the projection trick (3.38) to construct excited quasi-hole and quasi-particle states in these new filling fractions. For each, we can determine the charge and statistics. We won't do this here, but we will later revisit this question in Section 5.2.4 from the perspective of Chern-Simons theory.

3.3.3 The Half-Filled Landau Level

The composite fermion construction does a good job of explaining the observed plateaux. But arguably its greatest success lies in a region where no quantum Hall state is observed: $\nu = 1/2$. (Note that the Laughlin state for $m = 2$ describes bosons at half filling; here we are interested in the state of fermions at half filling). Looking at the data, there's no sign of a plateaux in the Hall conductivity at $\nu = 1/2$. In fact, there seems to be a distinct absence of Hall plateaux in this whole region. What's going on?!

The composite fermion picture gives a wonderful and surprising answer to this. Consider a composite fermion consisting of an electron bound to two vortices. If $\nu = 1/2$, so that the electrons have density $n = B/2\Phi_0$ then the effective magnetic field experienced by the composite fermions is (3.35)

$$B^\star = B - 2n\Phi_0 = 0 \tag{3.40}$$

According to this, the composite fermions shouldn't feel a magnetic field. That seems kind of miraculous. Looking at the data, we see that the $\nu = 1/2$ quantum Hall state occurs at a whopping $B \approx 25\ T$ or so. And yet this cartoon picture we've built up of composite fermions suggests that the electrons dress themselves with vortices so that they don't see any magnetic field at all.

So what happens to these fermions? Well, if they're on experiencing a magnetic field, then they must pile up and form a Fermi sea. The resulting state is simply the compressible state of a two-dimensional metal. The wavefunction describing a Fermi sea of non-interacting fermions is well known. If we have N particles, with position $\mathbf{r}_i$, and the N lowest momentum modes are $\mathbf{k}_i$, then we place particles in successive plane-wave states $e^{i\mathbf{k}_i\cdot\mathbf{r}_i}$ and subsequently anti-symmetrise over particles. The resulting slater determinant wavefunction is

$$\psi_{\text{Fermi Sea}} = \det\left(e^{i\mathbf{k}_i\cdot\mathbf{r}_j}\right) \tag{3.41}$$

The Fermi momentum is defined to be the highest momentum i.e. $k_F \equiv |\mathbf{k}_N|$. Once again, this isn't a lowest Landau level wavefunction since, in complex coordinates, $\mathbf{k}\cdot\mathbf{r} = \frac{1}{2}(k\bar{z} + \bar{k}z)$. This is cured, as before, by the projection operator giving us the ground state wavefunction at $\nu = 1/2$,

$$\tilde{\psi}_{\nu=\frac{1}{2}} = \mathcal{P}_{LLL}\left[\prod_{i<j}(z_i - z_j)^2\ \det\left(e^{i\mathbf{k}_m\cdot\mathbf{r}_l}\right)\right] \tag{3.42}$$

where, as before, the $(z_i - z_j)^2$ factor captures the fact that each composite fermion contains two vortices. This state, which describes an interacting Fermi sea, is sometimes called the *Rezayi-Read* wavefunction. (Be warned: we will also describe a different class of wavefunctions in Section 4.2.3 which are called Read-Rezayi states!). There is a standard theory, due to Landau, about what happens when you add interactions to a Fermi sea known as Fermi liquid theory. The various properties of the state at $\nu = 1/2$ and its excitations were studied in this context by Halperin, Lee and Read, and is usually referred to as the HLR theory[21].

[21]The paper is "*Theory of the half-filled Landau level*", Phys. Rev. B 47, 7312 (1993).

There is overwhelming experimental evidence that the $\nu = 1/2$ state is indeed a Fermi liquid. The simplest way to see this comes when we change the magnetic field slightly away from $\nu = 1/2$. Then the composite fermions will experience a very small magnetic field $B^\star$ as opposed to the original B. We can then see the Fermi surface and measure k_F through standard techniques such as de Haas-van Alphen oscillations. Perhaps the cleanest demonstration is then to look at excitations above the Fermi surface. Using simple classical physics, we expect that the particles will move in the usual cyclotron circles, with $x + iy = Re^{i\omega t}$ where $\omega = eB^\star/m^\star$. The slight problem here is that we don't know $m^\star$. But if we differentiate, we can relate the radius of the circle to the momentum of the particle which, in the present case, we can take to be $\hbar k_F$. We then get the simple prediction

$$R = \frac{\hbar k_F}{eB^\star}$$

which has been confirmed experimentally.

The Dipole Interpretation

Usually when we build a Fermi sea by filling successive momentum states, it's obvious where the momentum comes from. But not so here. The problem is that the electrons are sitting in the lowest Landau level where all kinetic energy is quenched. The entire Hamiltonian is governed only by the interactions between electrons,

$$H = V_{\rm int}(|\mathbf{r}_i - \mathbf{r}_j|)$$

Typically we take this to be the Coulomb repulsion (3.1) or some toy Hamiltonian of the kind described in Section 3.1.3. How can we get something resembling momentum out of such a set-up?

A potential answer comes from looking at the wavefunction (3.41) in more detail. The plane wave state is $e^{\frac{i}{2}(\bar{k}z + k\bar{z})}$. Upon making the substitution (3.39), this includes the term

$$\exp\left(ikl_B^2\frac{\partial}{\partial z}\right)$$

But this is simply a translation operator. It acts by shifting $z \to z + ikl_B^2$. It means that in this case we can rewrite the wavefunction (3.42) explicitly in holomorphic form,

$$\psi_{\nu=\frac{1}{2}} = \mathcal{A}\left[\prod_i e^{i\bar{k}_i z_i - |z_i|^2/4l_B^2}\right]\prod_{i<j}\Big((z_i + ik_i l_B^2) - (z_j + ik_j l_B^2)\Big)^2$$

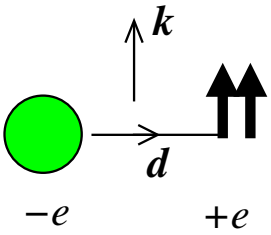


Figure 37: The composite fermion is a dipole like this.

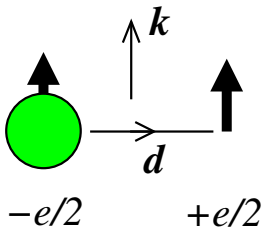


Figure 38: Or perhaps like this.

where $\mathcal{A}$ is what's left of the determinant, and means that we should anti-symmetrise over all different ways of pairing up k_i and z_i. Note that, for once, we've written the wavefunction including the exponential factor. The net result is that the zeros of the wavefunction — which are the vortices — are displaced by a distance $|kl_B^2|$ from the electron, in the direction perpendicular to $\vec{k}$.

As with much of the discussion on composite fermions, the ideas above are no more than suggestive. But they have turned out to be useful. Now that we have an extended object, thinking in terms of a reduced magnetic field is perhaps not so useful since the two ends can experience different magnetic fields. Instead, we can return to our original quasi-hole interpretation in which the vortices carry charge. One end then has a two vortices, each with charge $+e/2$. The other end consists of an electron with charge $-e$. The net result is the symmetric, dipole configuration shown in the figure with a dipole moment $\vec{d}$, with magnitude proportional to $\vec{k}$, such that $\vec{d}\cdot\vec{k}=0$ and $|\vec{d}| = ekl_B^2$.

The energy needed to produce such a dipole separation now comes entirely from the Coulomb interaction $V(|\mathbf{d}|)$ which we now interpret as $V(|\vec{k}|)$. On rotational grounds, the expansion of the potential energy should start with a term $\sim |\vec{d}|^2$ for small $\vec{d}$. This is the origin of the kinetic energy. The electron will drift along equipotentials of $V|\vec{k}|)$, while the vortices experience it as a magnetic field. The net effect is that both ends of the dipole move in the same direction, $\vec{k}$ with velocity $\partial V/\partial\vec{k}$ as expected.

We note that, more recently it's been suggested that it's better to think of the displacement as acting on just one of the two vortices bound to the electron[22]. This can be justified on the grounds that each electron always accompanies a single zero because of Pauli exclusion. The end with a single vortex has charge $+e/2$, while the end that consists of an electron bound to a single vortex has charge $-e+e/2=-e/2$. We get the same qualitative physics as before, but with $|\vec{d}| = ekl_B^2/2$ as shown in the

[22]This was proposed by Chong Wang and Senthil in "*Half-filled Landau level, topological insulator surfaces, and three dimensional quantum spin liquids*", arXiv:1507.08290.

figure. The only difference between these two possibilities lies in the Berry phase that the dipole acquires as it moves around the Fermi surface. This helps resolve an issue about particle-hole symmetry at half-filling which we will discuss briefly in Section 5.3.3.

3.3.4 Wavefunctions for Particles with Spin

Until now we've neglected the role of spin in the quantum Hall states, arguing that the Zeeman effect is sufficient to polarise the spin of the electron. Here we describe a simple generalisation of the Laughlin wavefunction for particles that carry spin[23].

We split our particles into two sets. The first set has spin-up, with positions $z_1, \ldots z_{N^\uparrow}$. The second set has spin-down, with positions $w_1, \ldots, w_{N^\downarrow}$. Note that each electron has a fixed spin which is an eigenvalue of S_z: we don't allow the spin to fluctuate, nor do we allow the spin to be misaligned from the z-axis. We'll relax this condition shortly.

If the two sets of particles didn't talk to each other, we can trivially take the product of two Laughlin wavefunctions,

$$\psi(z,w) = \prod_{i<j}^{N^\uparrow}(z_i - z_j)^{m_1} \prod_{k<l}^{N^\downarrow}(w_k - w_l)^{m_2}\, e^{-\sum |z_i|^2/4l_B^2 - \sum |\omega_i|^2/4l_B^2}$$

Such a state would have filling fraction $\nu^\uparrow = 1/m_1$ and $\nu^\downarrow = 1/m_2$, giving total filling fraction $\nu = \nu^\uparrow + \nu^\downarrow$.

Clearly there's nothing new in these wavefunctions. What's more, they miss the interesting physics. As we saw above, the Coulomb interactions are what drives the state to the Laughlin wavefunction. But these Coulomb interactions are blind to spin. They must also give correlations between the two sets of electrons. Halperin proposed to capture this with the simple wavefunction

$$\tilde{\psi}(z,w) = \prod_{i<j}^{N^\uparrow}(z_i - z_j)^{m_1} \prod_{k<l}^{N^\downarrow}(w_k - w_l)^{m_2} \prod_{i,k}(z_i - w_k)^n \tag{3.43}$$

where now $\tilde{\psi}$ means that we're dropping the exponential factors for both variables. This set of wavefunctions are characterised by the three integers and usually referred to as the (m_1, m_2, n) states, or sometimes as *Halperin states*.

[23]These wavefunctions were first introduced by Bert Halperin in "*Theory of the quantized Hall conductance*", Helv. Phys. Acta, 56 (1983).

These wavefunctions have very similar properties to the Laughlin states. In particular, the relative angular momentum is never less than m_1 between two spin-up particles, never less than m_2 for two down-spin particles and never less than n for particles of opposite spin. This kind of intuition allows us to build toy Hamiltonians, similar to those of Section 3.1.3, which have these wavefunctions as ground states.

Let's now compute the filling fractions of these wavefunctions. Following our calculation in Section 3.1.1, we'll look at the highest power of a given spin-up electron, say z_1. We see that this has maximum angular momentum $m_1 N^\uparrow + n N^\downarrow$ and hence fills out an area

$$A^\uparrow = 2\pi(m_1 N^\uparrow + n N^\downarrow) l_B^2$$

Meanwhile, the same computation for the spin-down particles gives us the area

$$A^\downarrow = 2\pi(m_2 N^\downarrow + n N^\uparrow) l_B^2$$

If we want to focus on the places where both spin-up and spin-down particles intermingle, we should take $A^\uparrow = A^\downarrow$. Clearly for a given state (m_1, m_2, n) this puts a constraint on, say, $N_\downarrow$ given $N_\uparrow$. The filling fractions are then

$$\nu^\uparrow = \frac{N^\uparrow}{m_1 N^\uparrow + n N^\downarrow} = \frac{m_2 - n}{m_1 m_2 - n^2}$$
$$\nu^\downarrow = \frac{N^\downarrow}{m_2 N^\downarrow + n N^\uparrow} = \frac{m_1 - n}{m_1 m_2 - n^2}$$

where, in the second equality, we have used the constraint that follows from choosing $A^\uparrow = A^\downarrow$. The total filling fraction is then

$$\nu = \nu^\uparrow + \nu^\downarrow = \frac{m_1 + m_2 - 2n}{m_1 m_2 - n^2} \tag{3.44}$$

The most prominent states of this kind have the form (m, m, n). These have filling fractions $\nu^\uparrow = \nu^\downarrow = \nu/2$ with

$$\nu = \frac{2}{m + n} \tag{3.45}$$

Interesting examples include

- $(3, 3, 1)$ with $\nu = 1/2$. Note that this is a genuine quantum Hall state at $\nu = 1/2$, as opposed to the Fermi liquid state described in Section 3.3.3. It has been seen in bi-layer samples, in which the z and w coordinate refer to the positions of particles in the two different layers[24].

[24]See Y. Suen et. al, "*Observation of a $\nu = 1/2$ Fractional Quantum Hall State in a Double-Layer Electron System*", Phys. Rev. Lett 68 9 (1992).

- $(3,3,2)$ with $\nu = 2/5$. This state competes with the spin-polarised Jain state that occurs at the same filling.

Given these states, we could now start to construct quasi-hole and quasi-particle states for these multi-component wavefunctions. The quasi-holes in the (m,m,n) state turn out to have charge $e/(m+n)$. We'll postpone this discussion to Section 5, where we'll see that we can describe both the (m_1,m_2,n) states and the Jain states of Section 3.3.2 in a unified framework.

Putting Spin Back In

So far, we've been calling the different sets of particles "spin-up" and "spin-down", but the wavefunctions (3.43) don't really carry the spin information. For example, there's no way to measure the spin of the particle in along the x-axis, as opposed to the z-axis. However, there's a simple way to remedy this. We just add the spin information, $\sigma = \uparrow$ or $\downarrow$ for each particle and subsequently anti-symmetrise (for fermions) over all $N = N_\uparrow + N_\downarrow$ particles. For (m,m,n) states, with $m > n$ and $N^\uparrow = N^\downarrow = N/2$, this is written as

$$\tilde{\psi}(z,\sigma) = \mathcal{A}\left[\prod_{i<j}^{N}(z_i - z_j)^n \prod_{1<i<j<N/2}(z_i - z_j)^{m-n} \prod_{N/2+1<k<l<N}(z_k - z_l)^{m-n} \,|\uparrow \ldots \uparrow\downarrow \ldots \downarrow\rangle\right]$$

where $\mathcal{A}$ stands for anti-symmetrise over all particles, exchanging both positions and spins. Since the spin state above is symmetric in the first $N/2$ spins and the second $N/2$ spins, we must have m odd. (For bosons we could symmetrise over all particles providing m is even).

A particularly interesting class of wavefunction are spin singlets. Given a bunch of N spins, one simple way to form a spin singlet state is to choose a pairing of particles — say (12) and (34) and so on — and, for each pair, forming the spin singlet

$$|12\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow_1\downarrow_2\rangle - |\downarrow_1\uparrow_2\rangle\Big)$$

Then the spin state $|\Psi\rangle = |12\rangle|34\rangle \ldots |N-1,N\rangle$ is a spin singlet.

An Aside: Of course, the spin singlet constructed above is not unique. The number of spin singlet states is given by the *Catalan number*, $N!/(N^\uparrow+1)!N^\uparrow!$ where $N = 2N^\uparrow$.

We now want to write a spin singlet quantum Hall wavefunction. (Note that this is the opposite limit to the Laughlin wavefunctions which were fully spin polarised). Since the spin singlet state is itself anti-symmetric, we now require, in addition to having m odd, that n is even. It is then straightforward to construct a spin singlet version of the $(n+1, n+1, n)$ Halperin state by writing

$$\tilde{\psi}(z, w, \sigma) = \mathcal{A}\left[\prod_{i<j}^{N}(z_i - z_j)^n \prod_{i<j\,\text{odd}} (z_i - z_j) \prod_{k<l\,\text{even}} (z_k - z_l)\ |12\rangle|34\rangle \ldots |N-1, N\rangle\right]$$

It can be seen to be a spin singlet because the last two factors are just Slater determinants for spin up and spin down respectively, which is guaranteed to form a spin singlet. Meanwhile, the first factor is a symmetric polynomial and doesn't change the spin. A stronger statement, which would require somewhat more group theory to prove, is that the $(n+1, n+1, n)$ Halperin states are the only spin singlets.

There is much more interesting physics in these quantum Hall states with spin. In particular, for the case $m = n$, the Halperin states become degenerate with others in which the spins do not lie in along the z-direction and the spin picks up its own dynamics. The resulting physics is much studied and associated to the phenomenon of quantum Hall ferromagnetism