Physics 8902: QFT 2. Lecture 1:

Reminder:

(1) ~~[illegible]~~. Euler-Lagrange equations and free scalar field

In QFT we deal with Lagrangians that define the action:

$$S = \int dt L = \int d^4x \, \mathcal{L}(x),$$

$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ - functional. Variation $\phi \to \phi + \delta\phi$ leads to $S = S_0 + \delta S$, where

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] =$$

$$= \int d^4x \left\{ \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi + \underbrace{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]}_{\text{Total deriv} \Rightarrow} \right\}$$

$\Rightarrow$ integration of the last term depends on the ~~spatial~~ values of the field at infinities. $\Rightarrow$ we have $A \partial_\mu B = -(\partial_\mu A) B$

in a Lagrangian.

Principle of the least action determines the equations of motion:

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

Example: $S = \int d^4x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V[\phi] \right] \Rightarrow$

$\Rightarrow$ Euler-Lagrange e/m are:

$$-V'[\phi] - \partial_\mu(\partial_\mu \phi) = 0, \quad \text{or}$$

$$\Box \phi + V'[\phi] = 0, \text{ where } \Box \equiv \partial_\mu^2 \text{ is}$$

the d'Alembertian.

When $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2$, the e/m are

$(\Box + m^2)\phi = 0$ - is called Klein-Gordon equation $\rightarrow$

$\rightarrow$ describes the equation of motion of the free scalar field.

(2) Second quantization: Consider real scalar theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2} m^2 \varphi^2.$$

Field/mode expansion is a representation of quantum fields as integrals over creation and annihilation operators of each momentum:

$$\Phi_0(\vec{x}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(\hat{a}_{\vec{k}} e^{i\vec{k}\vec{x}} + \hat{a}^+_{\vec{k}} e^{-i\vec{k}\vec{x}}\right),$$

where $\frac{1}{\sqrt{2\omega_k}}$ is introduced for convenience.

Here $\omega_k = \sqrt{\vec{k}^2 + m^2}$. Φ_0 – indicates it is a free field.

~~Fro~~

We see that if $[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}') \Rightarrow$

$$[\phi(\vec{x}), \phi(\vec{y})] = \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p 2\omega_q}} \left[(a_p e^{i\vec{p}\vec{x}} + a_p^+ e^{-i\vec{p}\vec{x}}), (a_q e^{i\vec{q}\vec{y}} + a_q^+ e^{-i\vec{q}\vec{y}})\right] =$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q 2\omega_p}} \left(e^{i\vec{p}\vec{x}} e^{-i\vec{q}\vec{y}} [a_p, a_q^+] + e^{-i\vec{p}\vec{x}} e^{i\vec{q}\vec{y}} [a_p^+, a_q]\right) = \left(\text{using } [a_k, a_k^+] = (2\pi)^3 \delta^{(3)}(k-k')\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p 2\omega_q}} \left[e^{i\vec{p}\vec{x}} e^{-i\vec{p}\vec{y}} - e^{-i\vec{p}\vec{x}} e^{i\vec{q}\vec{y}}\right] \times (2\pi)^3 \delta^3(\vec{p}-\vec{q}) =$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{2\omega_p}}_{\text{symmetric}} \underbrace{\left[e^{i\vec{p}(\vec{x}-\vec{y})} - e^{-i\vec{p}(\vec{x}-\vec{y})}\right]}_{\text{antisymmetric w.r.t. } \vec{p}\to-\vec{p}} = 0.$$

So we obtain $[\phi(\vec{x}), \phi(\vec{y})] = 0$ - canonical commutation relation for bosonic field.

③ Hamiltonian and canonical $\hat{\pi}$-field.

Hamiltonian is a functional of the fields and their conjugate momenta: $H = H[\hat{\phi}, \hat{\pi}]$.

The Lagrangian is the Legendre transform of the Hamiltonian

$$\mathcal{L}[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}]\, \dot{\phi} - H[\phi, \pi[\phi, \dot{\phi}]],$$

where $\pi[\phi, \dot{\phi}]$ is defined by $\dfrac{\partial H[\phi, \pi]}{\partial \pi} = \dot{\phi}$, or

$$\boxed{\pi = \frac{\partial \mathcal{L}[\phi, \dot{\phi}]}{\partial \dot{\phi}}}$$

For real scalar field theory:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - V[\phi] = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V[\phi],$$

wher $V[\phi]$ is the potential, we have

$\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$. For Hamiltonian we have

$$H = \pi \dot{\phi}[\phi, \pi] - \mathcal{L}[\phi, \dot{\phi}[\phi, \pi]] = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V[\phi].$$

Now for canonical $\hat{\pi}(\vec{x}) \equiv \dot{\hat{\phi}}[\vec{x}]$, we

have

$$\pi(\vec{x}) = \partial_t \phi(x)\Big|_{t=0} = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(\hat{a}_p e^{i\vec{p}\vec{x}} - \hat{a}_p^{+} e^{-i\vec{p}\vec{x}}\right),$$

where operator $\hat{\pi}$ is canonically conjugate to $\hat{\phi}$.

Hence we have

$$\left[\hat{\phi}(\vec{x}), \hat{\pi}(\vec{y})\right] = -i\int \frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\,\frac{1}{\sqrt{2\omega_q}}\times$$

$$\times\left(e^{i\vec{p}\vec{y}}e^{-i\vec{q}\vec{x}}\left[a_q^{+}, a_p\right] - e^{i\vec{q}\vec{x}}e^{-i\vec{p}\vec{y}}\left[a_q, a_p^{+}\right]\right) =$$

$$= \frac{i}{2}\int\frac{d^3p}{(2\pi)^3}\left[e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-i\vec{p}(\vec{x}-\vec{y})}\right] = \underline{\underline{i\,\delta^{(3)}(\vec{x}-\vec{y})}}$$

4 Fock space

normalization: Operator a_p^{+} creates a particle with momentum p

$$a_p^{+}|0\rangle = \frac{1}{\sqrt{2\omega_{\vec{p}}}}\,|\vec{p}\rangle,$$

where $|\vec{p}\rangle$ is a state with a single particle of momentum $\vec{p}$.

(5) Green's functions.

Consider a real scalar field-theory coupled to an external source:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} = \frac{1}{2}(\partial_\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2} m^2 \varphi^2 + J(x) \cdot \varphi.$$

Equation of motion corresponding to $\mathcal{L}$ reads:

$$\left[m^2 + \Box_{\vec{x}}\right] \varphi(\vec{x}) = J(\vec{x})$$

– is the Klein-Gordon eq. in the presence of an external source.

KG: $\left[m^2 + \Box_x\right] \varphi_0(\vec{x}) = 0.$

Claim: $\varphi(x) = \varphi_0(x) + i \int d^4x' \, J(x') \, G(x - x'),$

where $G(x-x')$ is the Green's function ~~sec~~ for the Klein-Gordon equation

$$(\Box_x + m^2) \, G(x - x') = -i \, \delta^{(4)}(x - x').$$

Proof: Calculate $\left(m^2 + \Box_x\right) \varphi(x) = \overbrace{\left(m^2 + \Box_x\right) \varphi_0(x)}^{= 0} +$

$$+ i \int d^4x' \, J(x') \left(m^2 + \Box_x\right) G(x - x') =$$

$$= \int d^4x' \, J(x') \, \delta^{(4)}(x - x') = J(x).$$

* The Feynman propagator:

Let us start with the free-field operator:

$$\phi_0(x,t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_k e^{-ikx} + a_k^+ e^{ikx} \right),$$

where $k_0 \equiv \omega_k = \sqrt{m^2 + \vec{k}^2}$. Next we consider the vacuum expectation value:

$$\langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \frac{1}{\sqrt{2\omega_{k_2}}}$$

$$\times \langle 0 | a_{k_1} a_{k_2}^+ | 0 \rangle e^{i(k_2 x_2 - k_1 x_1)}.$$

We have $\langle 0 | a_{k_1} a_{k_2}^+ | 0 \rangle = (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2) \Rightarrow$

$$\Rightarrow \langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k} e^{ik(x_2 - x_1)}.$$

Now, we are interested in

$$\langle 0 | T \{ \phi_0(x_1) \phi_0(x_2) \} | 0 \rangle =$$

$$= \langle 0 | \phi_0(x_1) \phi_0(x_2) | 0 \rangle \theta(t_1 - t_2) +$$

$$+ \langle 0 | \phi_0(x_2) \phi_0(x_1) | 0 \rangle \theta(t_2 - t_1) =$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big[e^{i\vec{k}(\vec{x}_1 - \vec{x}_2)} e^{-i\omega_k \tau} \theta(\tau) +$$

$$+ e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} e^{i\omega_k \tau} \theta(-\tau) \Big], \quad \underline{\tau = t_1 - t_2}.$$

Takin $k \to -k$ in the first term $\Rightarrow$

$$\Rightarrow \langle 0 | T\{\phi_0(x_1)\phi_0(x_2)\} | 0 \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{x}_1 - \vec{x}_2)} \times$$

$$\times \left[e^{i\omega_k \tau} \theta(-\tau) + e^{-i\omega_k \tau} \theta(\tau) \right].$$

Advanced retarded propagators.

Use the following identity:

$$e^{-i\omega_k \tau} \theta(\tau) + e^{i\omega_k \tau} \theta(-\tau) = \lim_{\varepsilon \to 0} \frac{-2\omega_k}{2\pi i} \times$$

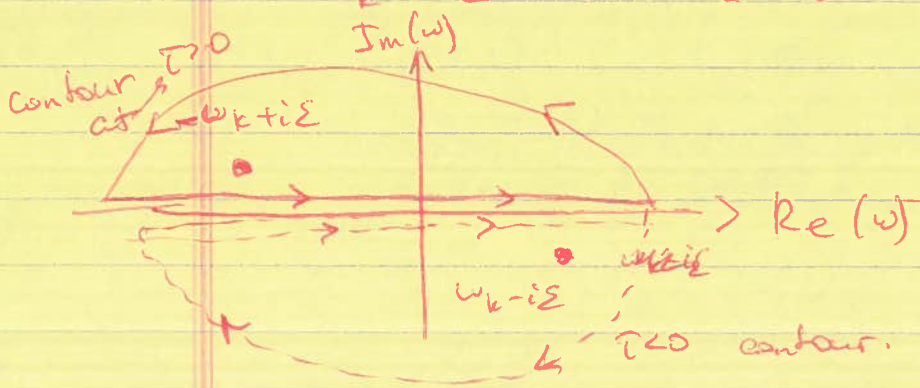
$$\times \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - \omega_k^2 + i\varepsilon} e^{i\omega\tau}.$$

Here

$$\frac{1}{\omega^2 - \omega_k^2 + i\varepsilon} = \frac{1}{[\omega - (\omega_k - i\varepsilon)][\omega - (-\omega_k + i\varepsilon)]} =$$

$$= \frac{1}{2\omega_k} \left[\frac{1}{\omega - (\omega_k - i\varepsilon)} - \frac{1}{\omega - (-\omega_k + i\varepsilon)} \right].$$

Here we work within $O(\varepsilon)^2$ accuracy and take $2\varepsilon\omega_k = \varepsilon$ as $\varepsilon \to 0$.



contour integral for Feynman propagator.

Performing contour integration, we arrive at

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega-(-\omega_k+i\varepsilon)} e^{i\omega\tau} = 2\pi i\, e^{-i\omega_k\tau}\,\theta(\tau) + O(\varepsilon).$$

Thus $$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \frac{d\omega}{\omega^2-\omega_k^2+i\varepsilon} e^{i\omega\tau} = -\frac{2\pi i}{2\omega_k}\left[e^{i\omega_k\tau}\,\theta(-\tau) + e^{-i\omega_k\tau}\,\theta(\tau)\right].$$

Putting together:

$$\langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\}|0\rangle = \lim_{\varepsilon\to 0}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\, e^{-i\vec{k}(\vec{x}_1-\vec{x}_2)} \times \int d\omega\, \frac{-2\omega_k}{2\pi i}\,\frac{1}{\omega^2-\omega_k^2+i\varepsilon}\, e^{i\omega\tau}$$

$$\Rightarrow\ G_F(x_1-x_2) = \langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\}|0\rangle =$$

$$= \int \frac{d^{(4)}k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\, e^{ik(x_1-x_2)}.$$

and $$\tilde{G}_F(p) = \frac{i}{p^2-m^2+i\varepsilon}$$

Reading for this Week:

[A] S-matrix: connects initial state $|in\rangle$ and final state $|out\rangle$ as:

$$|out, a\rangle = \hat{S}^{+} |in, a\rangle \Rightarrow$$

$$\Rightarrow \langle out, b | in, a\rangle = \langle in, b | \hat{S} | in, a\rangle .$$

$$T\{\phi(x_1)\ldots\phi(x_n)\} = \hat{S}^{-1}\, T\{\varphi_{in}(x_1)\ldots\varphi_{in}(x_n)\,\hat{S}\},$$

where $\hat{S} = T\exp\left\{-i\int_{-\infty}^{\infty} dt'\, \hat{H}_I(t')\right\}$

expressed via ϕ_{in}, π_{in}.

[B] Spectral representation of the Green's function.

Let $\Delta(x, x'; m) = \langle 0 | [\varphi_0(x)\,\varphi_0(x')] | 0\rangle$, then

$$\Delta(x, x') = \int_0^{\infty} d\sigma^2 \rho(\sigma^2)\, \Delta(x-x', \sigma)$$

$$= \underbrace{Z\,\Delta(x-x'; m)}_{\text{coherent part.}} + \underbrace{\int_{m^2}^{\infty} d\sigma^2 \rho(\sigma^2)\, \Delta(x-x'; \sigma)}_{\text{multiparticle continuum}}$$

where spectral density

$$\rho(q) = (2\pi)^3 \sum_N \delta^{(4)}(p_N - q) \left| \langle 0 | \varphi(0) | n \rangle \right|^2$$

$$\Rightarrow \quad \tilde{G}(p) = \frac{iZ}{p^2 - m^2 + i\varepsilon} + \int_{m^2}^{\infty} d\sigma^2 \rho(\sigma^2) \frac{i}{p^2 - \sigma^2 + i\varepsilon}$$

[c] Wick's theorem:

Normal ordering $\equiv$ all the a_p^+ operators are on the left of a_p operators.

Ex: $:(a_p^+ + a_p)(a_k^+ + a_k): = a_k^+ a_p + a_p^+ a_k + a_p a_k + a_p^+ a_k^+$.

No $\delta(p-k)$ - resulting from commutation relation should be accounted for.

$$\langle 0 | :\phi(x_1) \cdots \phi(x_n): | 0 \rangle = 0.$$

$$T\{\phi_0(x)\, \phi_0(y)\} = :\phi_0(x)\, \phi_0(y) + G_F(x,y):,$$

since $G_F(x,y) = \langle 0 | T\{\phi_0(x)\, \phi_0(y)\} | 0 \rangle$

and $\langle 0 | : \cdots : | 0 \rangle = 0$.

Wick's theorem:

$$T\left\{\phi_0(x_1)\ldots\phi_0(x_N)\right\} = :\phi_0(x_1)\ldots\phi_0(x_n):$$

$$+\sum_{\text{permut's}} :\phi_0(x_1)\ldots\underbrace{\overset{\text{take}}{\widehat{\phi(x_k)}}\ldots\overset{\text{out}}{\widehat{\phi(x_\ell)}}}\ldots\phi(x_n): \cancel{\times\langle 0|T\{\phi_0(x_k)\phi_0(x_\ell)\}}$$

$$\times\ \langle 0|T\{\phi_0(x_k)\,\phi_0(x_\ell)\}|0\rangle + \ldots +$$

$$+\underbrace{\sum \langle 0|T\{\phi_0(x_1)\,\phi_0(x_2)\}|0\rangle\ldots\langle 0|T\{\phi_0(x_{n-1})\,\phi_0(x_n)\}|0\rangle}_{\text{has no normal ordering} \Rightarrow}$$

$\Rightarrow$ if we calculate

$\langle 0|T\left[\phi_0(x_1)\ldots\phi(x_n)\right]|0\rangle$, the last term is the only surviving one.