1. ϕ^4 Interacting theory: Correlation functions.

We are interested in $H = H_0 + H_{int} =$

$$= H_{\text{Klein-Gordon}} + \int d^4x \, \frac{\lambda}{4!} \phi^4(x).$$

Ground state: $|\Omega\rangle$, which is generally different from $|0\rangle$ - the ground state of non-interacting theory.

Let us calculate the two-point Green's function:

$$G(x,y) = \langle \Omega | T \phi(x) \phi(y) | \Omega \rangle.$$

In the free theory, the Feynman propagator is:

$$G_F(x,y) = \langle 0 | T \phi_0(x) \phi_0(y) | 0 \rangle = G_F(x-y) =$$

$$= \int \frac{d^4p}{(2\pi)^4} \, \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i\varepsilon}.$$

How does this changes in interacting theory? Let us ~~do the~~ trace this change perturbatively in λ. λ - enters into definition of

H is λ dependent

$$\phi(x) = e^{iHt} \phi(\vec{x}) e^{-iHt} \quad \text{- Heisenberg picture.}$$

and ~~in~~ it enters into $|\Omega\rangle$

We have that at any fixed time t_0:

$$\phi(t_0, x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\vec{x}} + a_{\vec{p}}^{+}\, e^{-i\vec{p}\vec{x}} \right).$$

To obtain $\phi(t, \vec{x})$ at $t \neq t_0$, one can switch to the Heisenberg representation

$$\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x})\, e^{-iH(t-t_0)}.$$

Moreover, $H(\lambda = 0) \equiv H_0$, and thus

$$\phi(t, \vec{x})\Big|_{\lambda=0} = e^{iH_0(t-t_0)} \phi(t_0, \vec{x})\, e^{-iH_0(t-t_0)} = \phi_I(t, x),$$

where $\phi_I(t, \vec{x})$ is the field in <u>interaction</u> representation:

$$\phi_I(t, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ipx} + a_{\vec{p}}^{+}\, e^{ipx} \right)\Bigg|_{x_0 = t - t_0}$$

(this follows from diagonalization of H_0).

Formally, we have that

$$\phi(t, x) = e^{iH(t-t_0)} \underbrace{e^{-iH_0(t-t_0)} \phi_I(t, \vec{x})\, e^{iH_0(t-t_0)}}_{\phi(t_0, \vec{x})} e^{-iH(t-t_0)} \equiv$$

$$\equiv U^{+}(t, t_0)\, \phi_I(t, \vec{x})\, U(t, t_0),$$

where we have defined the unitary operator

$$U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$ — called

time-evolution operator.

Note that: $U(t_0,t_0) = 1$ and U is the solution of the following Schroedinger equation:

$$i\frac{\partial}{\partial t} U(t,t_0) = e^{iH_0(t-t_0)} \underbrace{(H-H_0)}_{H_{int}} e^{-iH_0(t-t_0)} =$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}_{H_I(t)} e^{iH_0(t-t_0)} e^{-iH(t-t_0)} =$$

$$= H_I(t)\, U(t,t_0)\text{, where}$$

$$H_I(t) = \int d^3x \frac{\lambda}{4!} \phi_I^4$$ — is the interaction Hamiltonian written in the interaction representation

Now, let us show that the solution to the Schroedinger eq

$$i\frac{\partial}{\partial t} U(t,t_0) = H_I(t)\, U(t,t_0)$$ is

given by the following series in λ:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2)$$

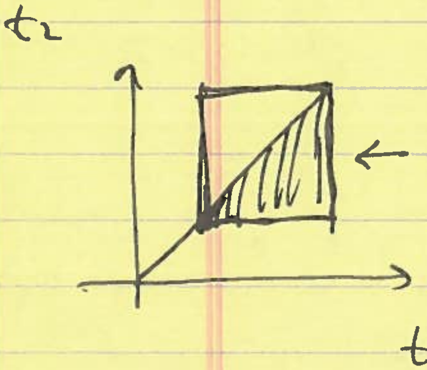
$$+ (-i)^3 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) + \ldots$$

It is straightforward to verify this:

(differentiating w.r.t t we obtain that each term gives the previous one $\times (-i) H_I(t) \Rightarrow$ the Schroedinger equation is satisfied).

Note the time-ordered products of H_I-operators

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\, T\{H_I(t_1) H_I(t_2)\}.$$



$\leftarrow \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2$, while $\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2$ covers the square.

and, similarly:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} H_I(t_1) \cdots H_I(t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1\, dt_2 \cdots dt_n \times$$
$$\times T\{H_I(t_1) \cdots H_I(t_n)\}.$$

Using these identities, we see that

$$U(t,t_0) = 1 + (-i) \int_{t_0}^{t} dt_1\, H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 dt_2\, T\{H_I(t_1) H_I(t_2)\} +$$
$$+ \cdots \equiv T\left\{\exp\left[-i \int_{t_0}^{t} dt'\, H_I(t')\right]\right\}.$$

Interestingly, the solution $U(t,t')$ satisfies the following identities

$$U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$$

$$U(t_1,t_3)\left[U(t_2,t_3)\right]^{+} = U(t_1,t_2).$$

Now let us discuss the ground state $|\Omega\rangle$ of the interacting theory, H: $E_0 \equiv \langle\Omega|H|\Omega\rangle$ is the ground state energy.

Let us start with the non-interacting ground state $|0\rangle$ and evolve through time with H:

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle,$$

where E_n are eigenenergies of H. Assume $\langle\Omega|0\rangle \neq 0$, i.e. there is some overlap. Then the above series contains $|\Omega\rangle$, and we can write

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle +$$

$$+ \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle.$$

Define the zero of energy as $H_0|0\rangle = 0$.

Since E_0 is the ground state energy, $E_n > E_0$ for all $n \neq 0$.

Then, we can get rid of all $n \neq 0$ terms in the series by sending

$$T \to \infty \cdot (1 - i\varepsilon).$$

In this case, the exponent e^{-iE_nT} dies slowest for $n=0$ and

$$|\Omega\rangle = \lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iE_0T} \langle \Omega | 0 \rangle\right)^{-1} e^{-iHT} |0\rangle$$

Since T is very large, we can shift it by constant t_0:

$$|\Omega\rangle = \lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iE_0(T+t_0)} \langle \Omega | 0 \rangle\right)^{-1} e^{-iH(T+t_0)} |0\rangle$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))} \langle \Omega | 0 \rangle\right)^{-1} e^{-iH(t_0-(-T))} \times$$

$$\times \underbrace{e^{-iH_0(-T-t_0)} |0\rangle}_{= |0\rangle \text{ as } H_0|0\rangle = 0.}$$

$$= \lim_{T \to \infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))} \langle \Omega | 0 \rangle\right)^{-1} U(t_0, -T) |0\rangle.$$

So we can obtain $|\Omega\rangle$ by evolving $|0\rangle$ from time $-T$ to t_0 with operator U.

Similarly:

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}.$$

Now let us return to the two-point Green's function. Assuming $x_0 > y_0 > t_0$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \times$$

$$\times \langle 0|U(T,t_0)\,\left[U(x_0,t_0)\right]^{+}\phi_I(x)\,U(x_0,t_0)\left[U(y_0,t_0)\right]^{+}\phi_I(y)$$

$$\times\, U(y_0,t_0)\;U(t_0,-T)\,|0\rangle\left(e^{-iE_0(t_0-(-T))}\cdot\langle\Omega|0\rangle\right)^{-1} =$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \times$$

$$\times\ \langle 0|U(T,x_0)\,\phi_I(x)\,U(x_0,y_0)\,\phi_I(y)\,U(y_0,-T)|0\rangle.$$

To deal with the factor in front, let us divide the whole expression by 1:

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \times$$

$$\times\ \langle 0|U(T,t_0)\,U(t_0,-T)|0\rangle.$$

Then, for $x_0 > y_0$, the expression for the Green's function becomes

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,x_0)\phi_I(x)U(x_0,y_0)\phi_I(y)U(y_0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

Notice that the fields in this expression are in time order. If we initially had $y_0 > x_0$, it would still work. Thus we arrive at

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\{\phi_I(x)\phi_I(y)e^{-i\int_{-T}^{T}dt\,H_I(t)}\}|0\rangle}{\langle 0|T\{e^{-i\int_{-T}^{T}dt\,H_I(t)}\}|0\rangle}$$

---

2. Wick's theorem

So the calculation of correlation functions / Green's functions in interacting theory is reduced to calculation of time-ordered products of free-field operators:

$$\langle 0|T\{\phi_I(x_1)\dots\phi_I(x_n)\}|0\rangle.$$

At $n=2$ we obtain the Feynman propagator.

For simplicity consider again

$\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle$. We will now generalize the calculation of this Feynman propagator to higher-point functions.

Decompose: $\phi_I(x) = \phi_I^+(x) + \phi_I^-(x)$ - is the sum of positive and negative frequency parts, where

$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p e^{-ipx}$$

$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p^+ e^{+ipx}.$$

This can be done for $\forall$ free field and

$$\phi_I^+(x)|0\rangle = 0 \quad , \langle 0|\phi_I^-(x) = 0.$$

For example, consider $x_0 > y_0$. $\Rightarrow$

$$\Rightarrow T\phi_I(x)\phi_I(y) \underset{x_0>y_0}{=} \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) +$$

$$+ \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) =$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) +$$

$$+ \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)].$$

Here, in all terms but in the commutator, all the a_p's are to the <u>right</u> of all a_p^+'s.

Example: $a_p^+ a_q^+ a_k a_\ell$,

Such an order is called "normal order" and it has vanishing VEV.

Example: $N(a_p a_k^+ a_q) \equiv a_k^+ a_p a_q$.

If we initially had $y_0 > x_0 \Rightarrow$ we would get the same normal ordered term with zero VEV, but the final commutator would be $[\phi_I^+(y)\ \phi_I^-(x)]$.

<u>Define:</u>

$$\overbracket{\phi(x)\,\phi(y)} \equiv \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for } x_0 > y_0 \\ [\phi^+(y), \phi^-(x)] & \text{for } y_0 > x_0 \end{cases}$$

Expectation value of this is Feynman propagator.

$$\overbracket{\phi_I(x)\,\phi_I(y)} = G_F(x-y)$$

We now have $T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbracket{\phi(x)\,\phi(y)}\}$

This can be generalized to

$$T\{\phi(x_1)\ldots\phi(x_n)\} = N\{\phi(x_1)\ldots\phi(x_n) + \text{all possible contractions}\}.$$

$\boxed{\text{Wick's theorem}}$

In the VEV, any term with uncontracted operators gives 0, since $\langle 0| N\{\text{any operator}\} |0\rangle \equiv 0$.

Example:

$$\langle 0| \phi_1 \phi_2 \phi_3 \phi_4 |0\rangle = G_F(x_1 - x_2)\, G_F(x_3 - x_4) + G_F(x_1 - x_3)\, G_F(x_2 - x_4) + G_F(x_1 - x_4)\, G_F(x_2 - x_3).$$