Radiative corrections / spectral representation.

* Field-Strength renormalization. (Residue of the Green's f-ion

Here we want to investigate two point correlation functions of interacting bosonic (ϕ) and fermionic (ψ) fields: $\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle$ and $\langle \Omega | T \psi(x) \psi(y) | \Omega \rangle$

In a free theory, $\langle 0 | T \phi(x) \phi(y) | 0 \rangle = G_F(x-y)$ is the amplitude for a particle to propagate from y to x.

What about interacting theories?

- we will not specify any interaction H_{int}
- we will not use perturbation theory
- we will consider scalar field. Similar results will aply for correlators of fields with spin and / or Dirac fields.

Translational invariancy $\Rightarrow$ $[\hat{P}, H] = 0$ and eigenstates of $\hat{P}$: $|\vec{p}\rangle$, ~~such that~~ are also eigenstates of $\hat{H}$. Let $|\lambda_0\rangle$ is an eigenstate of H with zero momentum: $\hat{H}|\lambda_0\rangle = m_\lambda|\lambda_0\rangle$, $\underline{\hat{P}|\lambda_0\rangle = 0}$

Then all boosts (Lorentz) of $|\lambda_0\rangle$, the $|\lambda p\rangle$ are eigenstates of H with all possible 3-momenta $\vec{p}$. Also $\hat{p}^\mu = (\hat{H}, \vec{\hat{P}})$.

So $|\lambda p\rangle$ are boosts of $|\lambda_0\rangle$ with momentum $\vec{p}$

We have that

$$|\vec p\rangle = \sqrt{2\omega_{\vec p}}\; a^+_{\vec p}\,|0\rangle \Rightarrow \langle \vec p|\vec q\rangle = 2\,\omega_p\,(2\pi)^3\,\delta^{(3)}(\vec p-\vec q).$$

A Lorentz transformation, $\Lambda \in O(1,3)$, will be implemented by a unitary operator $U(\Lambda)$

$$(*)\quad U(\Lambda)\,|\vec p\rangle = |\Lambda \vec p\rangle\,,$$ since we

require $\langle \Lambda\vec p|\Lambda\vec p\rangle = \langle \vec p|\vec p\rangle \Rightarrow$ Lorentz transform is a unitary operator.

From $(*) \Rightarrow$ $\sqrt{2\omega_{\Lambda\vec p}}\cdot a^+_{\Lambda\vec p} =$

$$= U(\Lambda)\sqrt{2\omega_{\vec p}}\; a^+_{\vec p}\, U^{-1}(\Lambda)$$

$$\Rightarrow\quad U(\Lambda)\, a^+_{\vec p}\, U^{-1}(\Lambda) = \sqrt{\frac{\omega_{\Lambda\vec p}}{\omega_{\vec p}}}\; a^+_{\Lambda\vec p}\,.$$

If we have an integral $I = \int \frac{d^4p}{(2\pi)^4} f(p)\cdot 2\pi$, it has to be constrained so that in satisfies the relativistic energy-momentum relation $\omega^2 = \vec p^{\,2} + m^2$.

We can use a δ-function to do this

$$I = \int \frac{d^4p}{(2\pi)^4}\cdot(2\pi)\, f(p)\,\delta(p^2 - m^2)\Big|_{p_0>0}.$$

$p_0 > 0$ means that relativistic energy is always ≥ 0. $\Rightarrow$ integration over p_0 is performed $\int_0^\infty dp_0$.

This can be written as

$$J = \int \frac{d^4 p}{(2\pi)^4} (2\pi) f(p)\, \delta(p^2 - m^2)\, \theta(p_0).$$

We have that $\delta[g(x) - g(x_0)] = \frac{1}{|g'(x_0)|} \delta(x) \Rightarrow$

$$\Rightarrow \delta\left((p_0)^2 - \vec{p}^{\,2} - m^2\right) = \frac{1}{2\sqrt{\vec{p}^{\,2} + m^2}}\, \delta(p_0) = \frac{1}{2\omega_{\vec{p}}}\, \delta(p_0),$$

where we took $\left\{ \begin{array}{l} g(x) = x^2 \\ x^2 = p_0^2 \\ x_0^2 = \vec{p}^{\,2} + m^2 \end{array} \right. \Big/ \Rightarrow \frac{1}{|g'(x_0)|} = \frac{1}{|2x_0|} = $

$$= \frac{1}{2\sqrt{\vec{p}^{\,2} + m^2}}.$$

Integration over p_0 now can be performed giving

$$J = \int \frac{d^4 p}{(2\pi)^4} (2\pi) f(p)\, \delta(p^2 - m^2) \Big|_{p_0 > 0} = \int \frac{d^3 p}{2\omega_{\vec{p}}} \cdot f(\omega_p, \vec{p}).$$

Taking $f \equiv 1$, we see that the integral

$I(f\equiv 1) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}}$ is a Lorentz invariant measure integral.

If $f = |\vec{p}\rangle\langle\vec{p}| \Rightarrow$ Lorentz invariant integral will be

$$\boxed{\int \frac{d^3p}{(2\pi)^3}\frac{|\vec{p}\rangle\langle\vec{p}|}{2\omega_{\vec{p}}} = \hat{1}}$$

- is the Lorentz invariant expansion of the unit operator. and the completeness relation for the single-particle states.

This can be verified as follows:

$$\langle\vec{p}|\vec{q}\rangle = \langle p|\int\frac{d^3p'}{(2\pi)^3}\frac{|\vec{p}'\rangle\langle\vec{p}'|}{2\omega_{\vec{p}'}}|\vec{q}\rangle =$$

$$= \langle\vec{p}|\int\frac{d^3p'}{(2\pi)^3}\frac{|\vec{p}'\rangle}{2\omega_{\vec{p}'}}\cdot\underbrace{\{2\omega_{\vec{p}'}(2\pi)^3\delta^{(3)}(\vec{p}'-\vec{q})\}}_{\langle\vec{p}'|\vec{q}\rangle}$$

$$= \underline{\langle\vec{p}|\vec{q}\rangle}.$$

Returning to page 2, where we assume $|\lambda p\rangle$ are relativistically normalized, where

$$\omega_{\vec{p}}(\lambda) \equiv \sqrt{|\vec{p}|^2+m_\lambda^2}\,,\text{ where } m_\lambda = h..$$

(see page 1).

Then the completeness relation of states $|\lambda p\rangle$ is

$$\hat{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}(\lambda)} |\lambda p\rangle\langle\lambda p|$$

where a) the sum runs over all zero-momentum states $|\lambda_0\rangle$.

b) the vacuum state $|\Omega\rangle$ has zero 3-momentum

Let us now insert this identity operator into the correlation function we would like to compute:

$$\langle\Omega| T\phi(x)\phi(y)|\Omega\rangle = \langle\Omega| T\phi(x)\,\hat{1}\,\phi(y)|\Omega\rangle =$$

Assuming $x_0 > y_0$, we will have

$$\langle\Omega|\phi(x)\phi(y)|0\rangle = \underbrace{\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle}_{\text{this is const} \to \text{ can be dropped.}} +$$

$$+\sum_\lambda \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}(\lambda)} \langle\Omega|\phi(x)|\lambda p\rangle\langle\lambda p|\phi(y)$$

The matrix element can be simplified to

$$\langle\Omega|\phi(x)|\lambda p\rangle = \langle\Omega| e^{i\hat{P}\cdot x}\phi(0)e^{-i\hat{P}x}|\lambda_{\vec{p}}\rangle$$

$$= \langle\Omega|\phi(0)|\lambda\vec{p}\rangle e^{-ipx}\Big|_{p_0=\omega_{\vec{p}}} =$$

$$= \langle \Omega | \phi(0) | \lambda_0 \rangle \, e^{-ipx} \Big|_{p_0 = E_p}$$

This follows from the fact that $\phi(0)$ and $|\Omega\rangle$ are Lorentz invariant, i.e.,

$$U \phi(0) U^{-1} = \phi(0) \quad \text{and} \quad U|\Omega\rangle = |\Omega\rangle, \; \langle \Omega | U^{-1} = \langle$$

Then $\langle \Omega | \phi(0) | \lambda_p \rangle = \langle \Omega | U^{-1} \phi(0) U | \lambda_p \rangle =$

$$= \langle \Omega | \phi(0) U | \lambda_p \rangle$$

and U can be such that $U_p | \lambda_p \rangle = | \lambda_0 \rangle$.

Finally, the expression for 2-point function acquires the form:

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle = \sum_\lambda \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\lambda^2 + i\varepsilon} e^{-ip(x-y)} \times$$

$$\times \left| \langle \Omega | \phi(0) | \lambda_0 \rangle \right|^2 .$$

Analogous expression works for $y_0 > x_0$. Therefore:

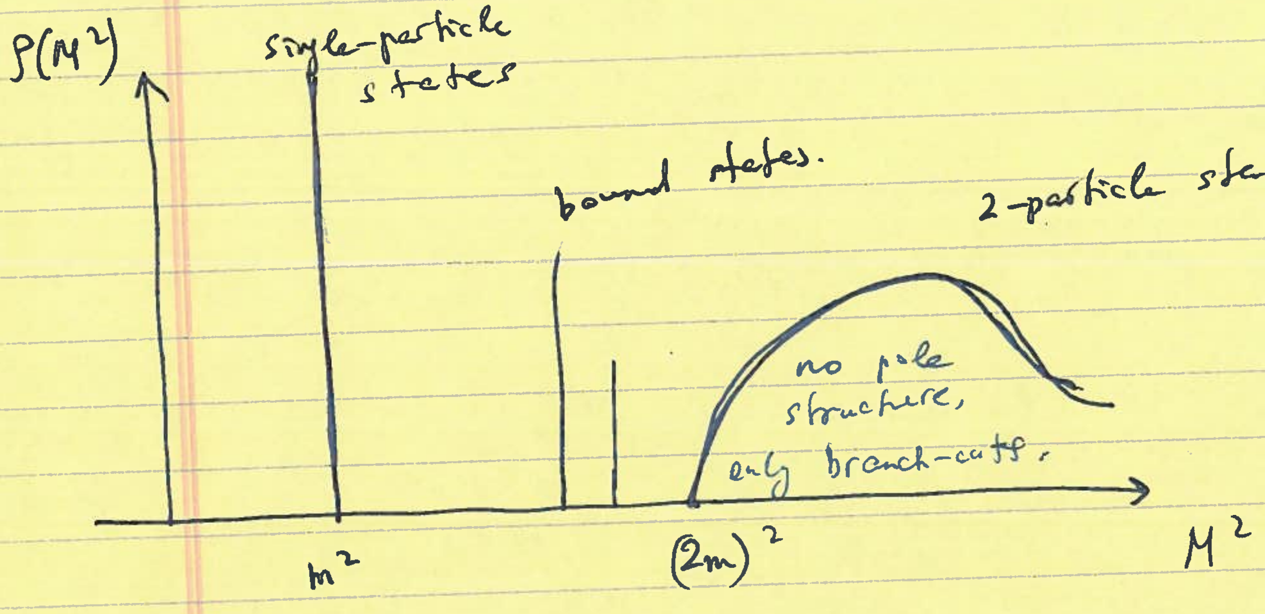
<u>Källen - Lehmann spectral representation:</u>

$$\langle \Omega | T \phi(x) \phi(y) | \Omega \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \, G_F(x-y, M^2),$$

where $\rho(M^2)$ is the spectral density function

$$\rho(M^2) \equiv \sum_\lambda (2\pi)\,\delta(M^2 - m_\lambda^2)\,|\langle\Omega|\phi(0)|\lambda_0\rangle|^2$$

Typically:



Single-particle states contribute δ-function:

$$\rho(M^2) = 2\pi\delta(M^2 - m^2)\cdot Z + f(M^2)\cdot\theta(M^2 - (2m)^2).$$

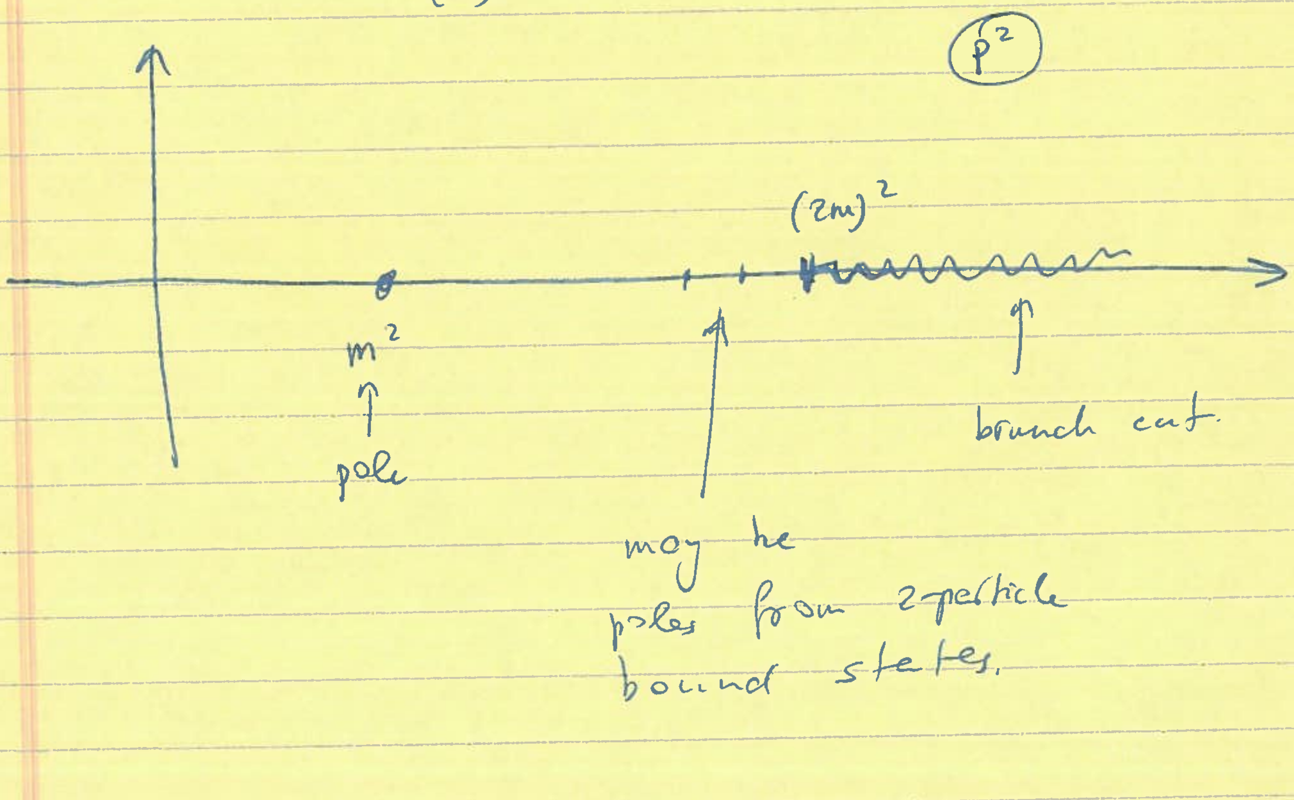
Z - called field-strength renormalization (pole residue).

This spectral decomposition yields the following form of the Fourier transform of the 2-point function:

$$\int d^4x\, e^{ipx} \langle \Omega | T\phi(x)\phi(y) | \Omega \rangle =$$

$$= \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) \underbrace{\frac{i}{p^2 - M^2 + i\varepsilon}}_{G_F(p)} =$$

$$= \frac{iZ}{p^2 - m^2 + i\varepsilon} + \int_{\sim(2m)^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\varepsilon}$$



Here $Z = |\langle \lambda_0 | \phi(0) | \Omega \rangle|^2$ is called field strength renormalization $\Rightarrow$ the probability for $\phi(0)$ to create a given zero-momentum state from the ground state (vacuum).

Quantum electrodynamics.

* Poincaré group.

The group of translations and Lorentz transformations is called Poincaré group, $ISO(1,3)$ $\Leftrightarrow$ the isometry group of Minkowski space.

If we rotate or boost to change the reference frame $\Rightarrow$ only the spin projections and the momenta will change (as determined by Poincaré group)

All other quantum numbers will remain the same

States transform under Poincare transformation as

$$|\psi\rangle \rightarrow P|\psi\rangle.$$

For a given representation of the group there is a basis of states s.t.

$$|\psi_i\rangle \mapsto P_{ij}|\psi_j\rangle.$$

If no subset of these states transform only among themselves, thee representation is irreducible.

The quantities we are computing in field theory are matrix elements

$M = \langle \psi_1 | \psi_2 \rangle$ => should be Poincaré invariant.

=> If $|\psi_1\rangle, |\psi_2\rangle$ transform covariantly under Poincaré P:

$$M = \langle \psi_1 | P^+ P | \psi_2 \rangle \Rightarrow P^+ P = \mathbb{1}$$

Particles transform under irreducible unitary representations of the Poincaré group.

Some of the representations of the Poincaré group are: the constant tensors, ϕ, V_μ, $T_{\mu\nu}$, ...

These are finite-D representations with 1, 4, 16, ... elements.

How to construct a unitary interacting theory of particles in these representations?

- we would like to embed the irreducible representations into objects with space-time indices.

- i.e. squeeze states of $S = 0, 1/2, 1, 3/2, 2,$ spin states into scalar fields $\phi(x)$, vector fields $V_\mu(x)$, tensor fields $T_{\mu\nu}(x)$, spinor fields $\psi(x)$.

The classical energy density: $\mathcal{E}$

$$\mathcal{E} = T_{00} = \sum_n \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} - \mathcal{L},$$

$E = \int d^3x\, \mathcal{E}$ - is the energy.

<u>S=0</u>: The Lagrangian is

$$\mathcal{L}(x) = \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) - \frac{1}{2} m^2 \phi^2(x)$$ - Lorentz invariant.

Eq. of motion: $(\Box + m^2)\, \phi(x) = 0$.

which has solutions $\phi = e^{\pm ipx}$, $p^2 = m^2$. $\Rightarrow$ has mass

Energy density: $\mathcal{E} = \frac{\partial \mathcal{L}}{\partial \dot\phi} \dot{\phi} - \mathcal{L} = \frac{1}{2} \left[(\partial_t \phi)^2 + (\vec\nabla \phi)^2 + m^2 \phi \right.$

is <u>positive definite</u>.

<u>S=1</u>: For $m > 1$ we must have $2S+1 = 3$ degrees freedom.

The smallest tensor-field we could embed these degrees of freedom in is a <u>vector field</u> A_μ, which has 4 components.

$4 = 3 \oplus 1 \iff$ 4D representation of Lorentz group is a direct sum of 3D $S=1$ rep and 1D $s=0$ representation of the rotation group $SO(3)$.

Try 1: write for massive $s=1$ field.

$$\mathcal{L} = -\frac{1}{2}\partial_\nu A_\mu \partial_\nu A_\mu + \frac{1}{2} m^2 A_\mu^2,$$

$$A_\mu^2 = A_\mu A^\mu.$$

$\Rightarrow$ Eq of motion are $(\Box + m^2) A_\mu = 0. \Rightarrow$

$\Rightarrow$ it has 4 propagating modes. The Lagrangian is a thus not for $s=1$ field but for 4 massive scalars A_0, A_1, A_2, A_3.

(or $4 = 1 \oplus 1 \oplus 1 \oplus 1$)

The energy density is:

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\mu)} \partial_t A_\mu - \mathcal{L} =$$

$$= -\frac{1}{2}\left[(\partial_t A_0)^2 + (\vec\nabla A_0)^2 + m^2 A_0^2\right] *$$

$$+ \frac{1}{2}\left[(\partial_t \vec A)^2 + (\nabla_i A_i)^2 + m^2 \vec A^2\right] \rightarrow$$

$\rightarrow$ it has $-$ sign for A_0-field and $+$ sign for $\vec A$-fields. $\Rightarrow$ we get some fields with negative energy $\Rightarrow$ unphysical. Switching the overall sign does not help.

Try 2: For massive $S=1$ theory we can have one more Lorentz-invariant two-derivative kinetic term in $(3+1)D$: $A_\mu \partial_\mu \partial_\nu A_\nu$.

Generally consider:

$$\mathcal{L} = \frac{a}{2} A_\mu \Box A_\mu + \frac{b}{2} A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2} m^2 A_\mu^2,$$

with constant a, b. To insure Lorentz invariance ~~transform~~ $(\partial_\nu A_\nu)$ ~~must~~ be Lorentz invariant $\Rightarrow$ A_μ transforms as a 4-vector. If A_μ transformed as 4 scalars, $(\partial_\mu A_\mu)$ would not be Lorentz invariant.

[Remember a scalar field under Lorentz transform $x \to \Lambda x$ transforms as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)$ and the derivative of the scalar field transforms as a vector $(\partial_\mu \phi)(x) \to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(y)$, $y = \Lambda^{-1} x$.].

The equations of motion are:

$$a \Box A_\mu + b\, \partial_\mu \partial_\nu A_\nu + m^2 A_\mu^2 = 0.$$

Taking ∂_μ gives: $\left[(a+b)\Box + m^2\right](\partial_\mu A_\mu) = 0.$

If $a = -b$, $m \neq 0 \Rightarrow$ E/M reduces to $\partial_\mu A_\mu = 0$, which removes one D.O.F.

Since $(\partial_\mu A_\mu) = 0$ is Lorentz invariant condition, it removes a complete representation, which with 1 D.O.F. can be only be the $s=0$ component.

Taking $a = -b = 1$:

$$\mathcal{L} = \frac{1}{2} A_\mu \Box A_\mu - \frac{1}{2} A_\mu \partial_\mu \partial_\nu A_\nu + \frac{1}{2} m A_\mu^2 =$$

$$= -\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m^2 A_\mu^2,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell tensor.

We see $F_{\mu\nu}$ appears just because of the constraint that the Lagrangian propagates $s=1$ particle and removes $s=0$ fields.

Energy-momentum tensor is:

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \partial_\nu A_\alpha - g_{\mu\nu} \mathcal{L} = - F_{\mu\alpha} \partial_\nu A_\alpha +$$

$$+ g_{\mu\nu} \left(\frac{1}{4} F_{\alpha\beta}^2 - \frac{1}{2} m^2 A_\alpha^2 \right).$$

Classically: $-\frac{1}{2} F_{\mu\nu}^2 = \frac{1}{2} \left(\vec{E}^2 - \vec{B}^2 \right)$, where

$\vec{E} = \partial_t \vec{A} - \vec{\nabla} A_0$ and $\vec{B} = \vec{\nabla} \times \vec{A}$.

Then: $\mathcal{E}$

$$\mathcal{E} = T_{00} = -(\partial_t A_\alpha - \partial_\alpha A_0)\partial_t A_\alpha + \frac{1}{2}\vec{B}^2 - \frac{1}{2}\vec{E}^2 - \frac{1}{2}m^2 A$$

$$= \frac{1}{2}(\vec{B}^2 + \vec{E}^2) + \partial_i A_0 (\partial_t A_0 - \partial_t A_0) -$$

$$- \frac{1}{2}m^2 A_0^2 + \frac{1}{2}m^2 \vec{A}^2$$

positive definite.

Finally: $$\mathcal{E} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \frac{1}{2}m^2 (A_0^2 + \vec{A}^2)$$

$$+ \underbrace{A_0 \partial_t (\partial_\mu A_\mu) - A_0 (\Box + m^2) A_0}_{\parallel} + \partial_i (A_0 F_{0i})$$

0 since $\underbrace{\partial_\mu A_\mu = 0, \quad (\Box + m^2) A_0 = 0}_{\text{equations of motion.}}$

$\partial_i (A_0 F_{0i})$ is a total spatial derivative $\Rightarrow$

$\Rightarrow$ does not contribute to total energy.

Solution of equations of motion:

Fourier transform $A_\mu(x)$. Since $(\Box + m^2) A_\mu = 0 \Rightarrow$

$$\Rightarrow A_\mu(x) = \sum_i \int \frac{d^3\vec{p}}{(2\pi)^3} \tilde{a}_i(\vec{p}) \, \varepsilon_\mu^i(p) \, e^{ipx}$$

$p_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$, for some basis vectors $\varepsilon_\mu^i(p)$.

We could take $i=1,\dots,4$ and use $\varepsilon_\mu^i(p)=\delta_\mu^i$. But we want $A_\mu(x)$ to satisfy

$$\partial_\mu A_\mu = 0 \quad \Rightarrow \quad p_\mu \varepsilon_\mu^i(p) = 0.$$

For $\forall$ fixed p_μ with $p^2=m^2 \Rightarrow$ there are three independent solutions by ~~three~~ 4-vectors $\varepsilon_\mu^i(p)$ - p-dependent.

$\varepsilon_\mu^i(p)$ are called polarization vectors.

Take p_μ to be in z-direction:

$p^\mu = (E, 0, 0, p_z)$, $E^2 - p_z^2 = m^2 \Rightarrow$ normalization

$\Rightarrow$ solutions of $p_\mu \varepsilon_\mu = 0$ and $\overbrace{\varepsilon_\mu^2 = -1 \;\; = \varepsilon_\mu^* \varepsilon_\mu}$

1) $\varepsilon_\mu^1 (0,1,0,0)$, $\varepsilon_\mu^2 = (0,0,1,0)$

$g = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$

These are transverse polarization

2) $\varepsilon_\mu^L = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right)$ - longitudinal polarization.

Massless spin-1:

$s=1$ theory in $m \to 0$ limit:

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 .}$$

However: a) $m^2 (\partial_\mu A_\mu) = 0$ - eq of motion is automatically satisfied $\Rightarrow$ no longer $\partial_\mu A_\mu = 0$.

b) Longitudinal polarization $\varepsilon_\mu^L = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m}\right) \to \infty$.

But let us just consider $\mathcal{L}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Check gauge invariance: $A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$, $\forall \alpha(x)$

Equation of motion: $\Box A_\mu - \partial_\mu(\partial_\nu A_\nu) = 0$ -
- 4 equation!

we need 3 eq only. separate 0 and $i=1,2,3$ components

$$\begin{cases} -\partial_j^2 A_0 + \partial_t \partial_j A_j = 0 \\ \Box A_i - \partial_i (\partial_t A_0 - \partial_j A_j) = 0 \end{cases}$$

Let us fix the gauge. ~~check that~~

$\partial_i A_i$ transforms under gauge transform as

$$\partial_i A_i \to \underbrace{\partial_i A_i}_{} + \partial_i^2 \alpha$$

Let us choose α so that $\partial_i A_i = 0$ — called Coulomb gauge.

$\Rightarrow$ $\partial_i^2 A_0 = 0$ from E/M.

Under Coulomb gauge:

$$\partial_i A_i \to \partial_i A_i + \partial_i^2 \alpha \Rightarrow A_\mu \to A_\mu + \partial_\mu \alpha$$

for $\forall$ α such that $\partial_i^2 \alpha = 0$.

Since $A_0 \to A_0 + \partial_t \alpha$ and $\partial_i^2 A_0 = 0$.

$\Rightarrow$ we have the symmetry to set $A_0 \equiv 0$.

$\Rightarrow$ E/M: $\Box A_i = 0$ $\quad i = 1, 2, 3$

<u>which propagates 3-modes.</u>